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Time Sub-Optimal Control of Triple Integrator

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Abstract: The time sub-optimal control is studied in this paper. The nonlinear controller that respects input saturations is derived for the simple linear system represented by the triple integrator. In comparison with the pure time optimal controller the designed sub-optimal controller changes its limit values smoothly with exponential behaviour. Similarly to the time optimal control the design is based on switching surfaces but these are shifted and modified according to the original ones in the time optimal control. This can assure the decrease of high sensitivity of time optimal control. New parameters introduced during the design correspond in linear cases to the poles of the closed loop system. They enable to tune the control changes. The time sub-optimal control is compared with model predictive control. The resulting formulas for the control value are complicated but they have an explicit form so they can be evaluated fast enough to be used in real time systems.

Keywords: time optimal control, input constraints, smooth switching, triple integrator.

1. INTRODUCTION

The optimality principle played an always an important role in the design of control circuits. In the previous century the optimal control was studied heavily in 50-ties and 60-ties (Athans and Falb, 1996). The real applications have shown that it is very sensitive to unmodelled dynamics, parametric variations, disturbances and noise. Therefore it was in the main-stream control strategies replaced by other techniques, e.g. pole assignment control. This allowed to choose the position of poles and so influence the speed of changes what enabled to decrease the sensitivity of a control circuit. This paper shows how it is possible to combine "fast" time optimal control with "slow" pole assignment control.

Generally the time optimal control can be solved by computation of switching surfaces. There are several ways how to derive them. They can result from the Pontryagin's maximum principle. Pavlov solved switching surfaces for the systems up to the third order from the phase trajectories (Pavlov, 1966). Switching surfaces can also be expressed by the set of algebraic equations (Walther et al., 2001) that results from the time solution in the phase space. Of course, for higher order systems it can be rather complicated to find the exact solution of such a set. This paper shows how it is possible for a simple linear system represented by the triple integrator to derive and solve a set of polynomial equations in order to get the control value in the exact form.

The controller proposed in this paper is not pure time optimal control. As it has been mentioned above it tries to combine qualities of both time optimal and pole assignment control (Huba, 2006). The time optimal control belongs to the nonlinear class of controllers whereas the pole assignment control is typically linear type of control. Switching between these two classes of control is assured by the saturation function that is applied individually to each mode of control. In this paper a set of additional parameters is introduced that corresponds to the set of poles of the closed loop in the linear case. These parameters specify exponential changes from one limit control value to the opposite one. By introducing such parameters in the control design the original switching surfaces valid for the time optimal control are modified. They are not smooth and consist of several regions. Then it is difficult to identify the corresponding region for an initial state. After deciding for the right region the control law is calculated according to the position of the representative point expressing the actual state with respect to the corresponding region of the switching surface.

The paper is organized in five chapters. After introduction and problem statement chapters there is the main chapter where the design of the sub-optimal controller is described in details. This chapter discusses the nonlinear dynamics decomposition and regions of the switching surface. There is a corresponding control law derived for each region. The fourth chapter shows time responses of the designed controller and compares it to the time optimal controller and model predictive controller. The paper is finished with short conclusions.

2. PROBLEM STATMENT

Let us consider the linear system given in the state space

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}^t\mathbf{x}\end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}^t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{1}$$

that represents the triple integrator. The control input signal is constrained

$$u = \langle U_1 \quad U_2 \rangle \tag{2}$$

The task is to design the time sub-optimal controller what means to drive the system from an initial state $\mathbf{x} = [x \quad y \quad z]^t$ to the desired state $\mathbf{x}_w$ in a minimum time $t_{\min}$ under the additional condition that limits the changes of the control action between two opposite values. When it is required that these changes should have an exponential behaviour the additional condition can be expressed by a scalar function $\varsigma_i : R^n \to R$ representing the distance of the current state $\mathbf{x}$ from the switching surface (curve, point) and it holds

$$\frac{d\varsigma_i}{dt} = \alpha_i \varsigma_i \ , \ \alpha_i \in R^- \ , \ i = 1,2,3 \tag{3}$$

For the sake of simplicity we should admit that using a coordinate transformation it is always possible to set the desired state equal to the origin $\mathbf{x}_w = \mathbf{0}$.

3. TIME SUB-OPTIMAL CONTROLLER DESIGN

It is well known that minimum time optimal control with saturated input leads to the control action with at most n intervals switching between limit values where n represents the order of the system. Usually the control algorithm results in deriving switching surfaces as functions of states which signs determine the switching times. It can be very hard task to express these functions exactly and there is no general solution for higher order systems ($n > 3$). Bang-bang control in practice is not desirable because of chattering and noise effects but there are techniques have to cope with them (Pao and Franklin, 1993, Bistak et al. 2005).

The presented sub-optimal controller design belongs to one of them. This time the control action will not be calculated as the sign of the switching surface but will result from (3). If we apply the condition (3) also for the switching curve and switching point this will influence the construction of the switching surface itself. We will explain it with the help of a state vector nonlinear decomposition.

3.1 Nonlinear Decomposition

Let us consider ordered coefficients

$$\alpha_3 < \alpha_2 < \alpha_1 < 0 \tag{4}$$

Then the eigenvectors

$$\mathbf{v}_i = [\alpha_i \mathbf{I} - \mathbf{A}]^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{\alpha_i^3} & \frac{1}{\alpha_i^2} & \frac{1}{\alpha_i} \end{bmatrix}^t \tag{5}$$

form a base of the state space. In the linear case any point of the state space can be expressed as

$$\mathbf{x} = q_1\mathbf{v}_1 + q_2\mathbf{v}_2 + q_3\mathbf{v}_3 \ , \ q_1, q_2, q_3 \in R \tag{6}$$

Because the control signal is limited only the points where $u = \sum_{i=1}^{3} q_i$ fulfils (2) are covered by (6). In order to express the whole space we have to introduce the nonlinear decomposition of the state

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 \tag{7}$$

where each mode $\mathbf{x}_i$

$$\mathbf{x}_i = e^{\mathbf{A}t_i} q_i \mathbf{v}_i + \int_0^{t_i} e^{\mathbf{A}(t-\tau)} \mathbf{b} q_i(\tau) d\tau \tag{8}$$

consists of a linear part given by the parameter q_i

$$q_i \in \left\langle U_1 - \sum_{k=1}^{i-1} q_k \quad U_2 - \sum_{k=1}^{i-1} q_k \right\rangle \quad when \quad t_i = 0 \tag{9}$$

and a nonlinear part specified by the parameter t_i

$$0 < t_i < t_{i-1} \quad when \quad q_i = U_{3-j} - \sum_{k=1}^{i-1} q_k \ , \ t_0 = \infty \ , \ j = 1,2 \tag{10}$$

After substituting (1) and (5) into (8) one gets

$$\mathbf{x}_i = \begin{bmatrix} \frac{1}{\alpha_i^3} - \frac{t_i}{\alpha_i^2} + \frac{t_i^2}{2\alpha_i} - \frac{t_i^3}{6} \\ \frac{1}{\alpha_i^2} - \frac{t_i}{\alpha_i} + \frac{t_i^2}{2} \\ \frac{1}{\alpha_i} - t_i \end{bmatrix} q_i \tag{11}$$

If we take the subsystem $\mathbf{x}_1$ it represents a one-dimensional variety that corresponds to the switching curve. Points from the linear part of $\mathbf{x}_1$ where (9) is fulfilled satisfy (3), i.e. they are decreasing the distance ς_1 from the origin. In this case the system is moving along the line. The other points of the subsystem $\mathbf{x}_1$ given by (10) could not fulfil (3) because of the limited control value (2). They are approaching the linear part of $\mathbf{x}_1$ with the limit control value so they are moving along the trajectory in the form of a curve. In this case (3) is superimposed by (2).

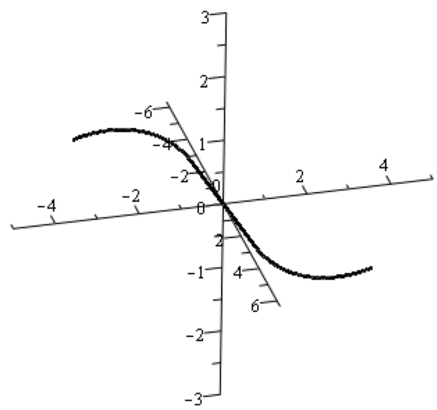

Fig. 1. Subsystem $\mathbf{x}_1$ representing the switching curve

Similarly we can create a two-dimensional variety that will express the switching surface. We simply add to the subsystem $\mathbf{x}_1$ the subsystem $\mathbf{x}_2$. The points of the subsystem $\mathbf{x}_1$ become the target points for the second subsystem $\mathbf{x}_2$.

$$\mathbf{x}_{12} = \mathbf{x}_1 + \mathbf{x}_2 \tag{12}$$

This time we define the distance ς_2 in the direction of the second eigenvector $\mathbf{v}_2$. The points of the $\mathbf{x}_2$ try to reach the $\mathbf{x}_1$ points according to (3) if it does not break (2). Otherwise they are moving with the limit control value q_2 given by (10).

Again there exist a linear and a nonlinear parts of the subsystem $\mathbf{x}_2$. In combination with the previous subsystem $\mathbf{x}_1$ we get four possibilities, i.e. four regions of the switching surface with respect to the limit and nonzero values of q_1, q_2, t_1, t_2 (Fig. 2). If we take into account the parameter $j = 1{,}2$ the number of the regions doubles. Later on we will describe these regions in details and derive for each of them the corresponding control value.

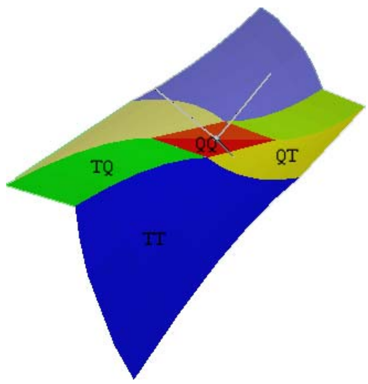

Fig. 2. Regions of the switching surface (denoted for *j*=1)

To cover the whole space we should realize also the third subsystem $\mathbf{x}_3$ but in the presented control algorithm design it is not necessary. To give reasonable results that can be applied in real time applications we simplified the third subsystem to following one

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} q_3 \quad , \quad q_3 \in R \tag{13}$$

It represents the unit vector in the direction of the x-axis multiplied by the quotient q_3. Thus the quotient q_3 expresses the distance ς_3 between the current state and the switching surface that is measured in the direction of the x-axis. This simplification enables easier localization of the initial state with respect to the regions of the switching surface because it represents the projection of the switching surface to the (y,z)-plane where the borders between regions are parabolic curves or lines.

In (Ťapák et al., 2006) one can find the solution of the control algorithm when the third subsystem was given by the third eigenvector $\mathbf{v}_3$ multiplied by the quotient q_3

$$\mathbf{x}_3 = q_3 \mathbf{v}_3, q_3 \in R \tag{14}$$

but this was not in the form suitable for real time systems.

After completely decomposing the system to the three subsystems (7) it is necessary to derive the formula of the corresponding region of the switching surface. This comes from the set of equations (12) when the parameters t_i or q_i are evaluated from the last two equations and replaced in the first one. Then one gets the formula for the corresponding switching surface in the form

$$x = f(y,z) \;\; ; \;\; f : R^2 \to R \tag{15}$$

Now the resulting control value can be computed from (3) when we realize that the distance $\varsigma_3 = q_3$ can be expressed as the difference between the x-coordinate of the initial point and the x-coordinate of the switching surface given by (15) in the form $f(y,z)$

$$\varsigma_3 = x - f(y,z) \tag{16}$$

After substituting (16) into (3) and taking into account (1) it results in

$$\frac{d\varsigma_3}{dt} = \frac{dx}{dt} - \frac{df(y,z)}{dt} = y - \frac{df(y,z)}{dy} z - \frac{df(y,z)}{dz} u = \alpha_3 (x - f(y,z)) \tag{17}$$

Finally the control value u can be isolated

$$u = \frac{-\alpha_3 (x - f(y,z)) + y - \dfrac{df(y,z)}{dy} z}{\dfrac{df(y,z)}{dz}} \tag{18}$$

The resulting control value u must be limited by (2).

As one can see from (18) the only one term not evaluated yet is $f(y,z)$ representing the switching trajectory. Because it differs according to the regions of the switching trajectory we will evaluate it individually.

3.2 Control for Region QQ

The region QQ denotes the subset of (12) where both subsystems $\mathbf{x}_1$ and $\mathbf{x}_2$ are in the linear cases, i.e. (9) is fulfilled for q_1 and q_2. The parameters q_1 and q_2 can be evaluated from the last two equations of the set (12). After using (11) and substituting $t_1 = t_2 = 0$ into (12) one gets

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{q_1}{\alpha_1^3} + \dfrac{q_2}{\alpha_2^3} \\ \dfrac{q_1}{\alpha_1^2} + \dfrac{q_2}{\alpha_2^2} \\ \dfrac{q_1}{\alpha_1} + \dfrac{q_2}{\alpha_2} \end{bmatrix} \tag{19}$$

And for parameters q_1 and q_2 it yields

$$q_1 = -\frac{\alpha_1^2 (z - y\alpha_2)}{\alpha_2 - \alpha_1} \tag{20}$$

$$q_2 = \frac{\alpha_2^2 (-\alpha_1 y + z)}{\alpha_2 - \alpha_1} \tag{21}$$

By the substitution of (20) and (21) into the first equation of (19) we derive the analytical expression for the region QQ

$$x = f(y,z) = -\frac{z - y\alpha_2}{\alpha_1(\alpha_2-\alpha_1)} + \frac{-\alpha_1 y + z}{\alpha_2(\alpha_2-\alpha_1)} \tag{22}$$

According to (18) the control value u results in the form

$$\begin{aligned} u = &-y\alpha_1\alpha_2 + \alpha_2 z + \alpha_1 z + \alpha_3 x\alpha_1\alpha_2 \\ &-\alpha_3 y\alpha_2 - \alpha_3\alpha_1 y + \alpha_3 z \end{aligned} \tag{23}$$

This is the well-known linear pole assignment controller for the triple integrator.

3.3 Control for Region TQ

By TQ we denote the region of the switching surface when the subsystem $\mathbf{x}_1$ is in the nonlinear cases, i.e. for its states (10) is valid and the subsystem $\mathbf{x}_2$ is in the linear cases, i.e. (9) comes true. The procedure how to derive the control value is very similar to that one performed in the previous region QQ. First we express (12) when $q_1 = U_j$ and $t_2 = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} U_j\left(\frac{1}{\alpha_1^3} - \frac{t_1}{\alpha_1^2} + \frac{1}{2}\frac{t_1^2}{\alpha_1}\right) - \frac{1}{6}U_j t_1^3 + \frac{q_2}{\alpha_2^3} \\ U_j\left(\frac{1}{\alpha_1^2} - \frac{t_1}{\alpha_1}\right) + \frac{1}{2}U_j t_1^2 + \frac{q_2}{\alpha_2^2} \\ \frac{U_j}{\alpha_1} - U_j t_1 + \frac{q_2}{\alpha_2} \end{bmatrix} \tag{24}$$

For this and following calculations we have used the Maple computer algebra system and because of the complexity of several expressions we have used the Maple outputs.

Now it is necessary to solve the last two equations of the set (24). The difference consists in that the second equation of the set (24) is now the quadratic equation. From its two solutions we have chosen such one that assures the positive value of t_2. Then after introducing the notation for the discriminant DTQ

$$DQT = -2\alpha_1^2 U_j z\alpha_2 + \alpha_1^2 U_j^2 - U_j^2\alpha_2^2 + 2y\alpha_1^2\alpha_2^2 U_j \tag{25}$$

the parameters q_2 and t_1 it can be expressed

$$q_2 = \frac{z\alpha_1\alpha_2 - \alpha_1 U_j + sign(U_j)\sqrt{DTQ}}{(U_{3-j} - U_j)\alpha_1} \tag{26}$$

$$t_1 = -\frac{-U_j\alpha_2 + \alpha_1 U_j - sign(U_j)\sqrt{DTQ}}{U_j\alpha_1\alpha_2} \tag{27}$$

Again we substitute (26) and (27) into the first equation of (24) and get the expression for the region TQ

$$\begin{aligned} x = \frac{1}{6}\frac{1}{U_j^2\alpha_1^3\alpha_2^3}\Big(&2U_j^3\alpha_2^3 + 3U_j^3\alpha_2^2\alpha_1 - 3U_j^2 \\ &\alpha_2^2 sign(U_j)\sqrt{DTQ} - 5\alpha_1^3 U_j^3 + 3\alpha_1^2 U_j^2 sign(U_j)\sqrt{DTQ} \\ &+ 3\alpha_1 U_j sign(U_j)^2 DTQ - sign(U_j) DTQ^{3/2} + 6\alpha_1^3 U_j^2 z\alpha_2\Big) \end{aligned} \tag{28}$$

From (18) the control value u is

$$\begin{aligned} u = \frac{1}{3}\frac{1}{\alpha_2^2\alpha_1^4(-y\alpha_2 + z)}\Big(&-3\alpha_2^4\alpha_1^4 zy + 2\alpha_3 U_j^2\alpha_2^2\alpha_1^2 - \alpha_3 U_j \\ &\alpha_2^3\alpha_1^2 z + \alpha_3 U_j\alpha_2^4 y\alpha_1^2 + \alpha_3\alpha_1^4 U_j z\alpha_2 - \alpha_3\alpha_1^4 U_j y\alpha_2^2 + 2\alpha_3 \\ &\alpha_1^4 z^2\alpha_2^2 - \alpha_3 U_j^2\alpha_2^4 - \alpha_3\alpha_1^4 U_j^2 - 3y\, sign(U_j)\sqrt{DTQ}\,\alpha_1^3\alpha_2^3 \\ &-\alpha_3 sign(U_j)\sqrt{DTQ}\,U_j\alpha_2^3 + \alpha_3 sign(U_j)\sqrt{DTQ}\,\alpha_1^3 U_j \\ &-4\alpha_3\alpha_1^4 z\alpha_2^3 y + 2\alpha_3 y^2\alpha_1^4\alpha_2^4 + 3\alpha_2^2\alpha_1^3 z\, sign(U_j)\sqrt{DTQ} \\ &-3\alpha_3 sign(U_j)\sqrt{DTQ}\,\alpha_1^3 y\alpha_2^2 + 3\alpha_3 sign(U_j)\sqrt{DTQ}\,x\alpha_1^3 \\ &\alpha_2^3 + 3\alpha_2^3\alpha_1^4 z^2\Big) \end{aligned} \tag{29}$$

3.4 Control for Region TT

The region TT denotes the subset of (12) where both subsystems $\mathbf{x}_1$ and $\mathbf{x}_2$ are in the nonlinear cases, i.e. (10) is fulfilled for t_1 and t_2. Again the parameters t_1 and t_2 can be computed from the last two equations of the (12). This time we substitute $q_1 = U_j$ and $q_2 = U_{3-j} - U_j$ into (12)

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \Bigg[\Bigg[&U_j\left(\frac{1}{\alpha_1^3} - \frac{t_1}{\alpha_1^2} + \frac{1}{2}\frac{t_1^2}{\alpha_1}\right) - \frac{1}{6}U_j t_1^3 + (U_{3-j} \\ &- U_j)\left(\frac{1}{\alpha_2^3} - \frac{t_2}{\alpha_2^2} + \frac{1}{2}\frac{t_2^2}{\alpha_2}\right) - \left(\frac{1}{6}U_{3-j} - \frac{1}{6}U_j\right)t_2^3\Bigg], \\ &\left[U_j\left(\frac{1}{\alpha_1^2} - \frac{t_1}{\alpha_1}\right) + \frac{1}{2}U_j t_1^2 + (U_{3-j} - U_j)\left(\frac{1}{\alpha_2^2} - \frac{t_2}{\alpha_2}\right) \right. \\ &\left. + \left(\frac{1}{2}U_{3-j} - \frac{1}{2}U_j\right)t_2^2\right], \\ &\left[\frac{U_j}{\alpha_1} - U_j t_1 + \frac{U_{3-j} - U_j}{\alpha_2} - (U_{3-j} - U_j)t_2\right]\Bigg] \end{aligned} \tag{30}$$

and when solving the last two equations again the criterion for the choice of the right solution is that the times t_1 and t_2 must be positive.

$$\begin{aligned} DTT = &-\alpha_2^2\alpha_1^2 U_{3-j} z^2 U_j + \alpha_2^2 U_j^2 z^2\alpha_1^2 - \alpha_1^2 U_{3-j}^3 U_j + 2U_j^2\alpha_1^2 \\ &U_{3-j}^2 + 2U_{3-j}^2 y\alpha_1^2\alpha_2^2 U_j - U_{3-j}^2 U_j^2\alpha_2^2 - 2U_{3-j}U_j^2 y\alpha_1^2 \\ &\alpha_2^2 + U_{3-j}U_j^3\alpha_2^2 - U_{3-j}U_j^3\alpha_1^2 \end{aligned} \tag{31}$$

$$t_1 = \frac{-U_j U_{3-j}\alpha_2 + \alpha_1 U_j z\alpha_2 + \sqrt{DTT}}{U_j\alpha_1 U_{3-j}\alpha_2} \tag{32}$$

$$\begin{aligned} t_2 \\ = &\frac{1}{\alpha_2\alpha_1 U_{3-j}(U_{3-j} - U_j)}\Big(\alpha_1 U_{3-j} z\alpha_2 - \alpha_1 U_j z\alpha_2 \\ &+ U_j\alpha_1 U_{3-j} - \alpha_1 U_{3-j}^2 - \sqrt{DTT}\Big) \end{aligned} \tag{33}$$

After using (32) and (33) in the first equation of the set (30) the points of the region TT can be expressed

$$\begin{aligned}
x = {} & \frac{1}{6}\,\frac{1}{U_{3-j}^2\,\alpha_2^3\,\alpha_1^3\,U_j^2\,\left(-U_{3-j}+U_j\right)^2}\Big(-2\,\alpha_2^3\,U_j^3\,z^3\,\alpha_1^3\,U_{3-j} + \\
& \alpha_2^3\,U_j^4\,z^3\,\alpha_1^3 + 3\,\alpha_2^3\,U_j^3\,U_{3-j}^3\,z\,\alpha_1 - 6\,\alpha_2^3\,U_j^4\,U_{3-j}^2\,z\,\alpha_1 + 3\,\alpha_2^3 \\
& U_j^5\,U_{3-j}\,z\,\alpha_1 - 3\,U_j^5\,\alpha_2\,z\,\alpha_1^3\,U_{3-j} + U_j^2\,\alpha_2^3\,\alpha_1^3\,U_{3-j}^2\,z^3 + 3 \\
& U_j^2\,\alpha_2\,\alpha_1^3\,U_{3-j}^4\,z - 9\,U_j^3\,\alpha_2\,\alpha_1^3\,U_{3-j}^3\,z + 9\,U_j^4\,\alpha_2\,\alpha_1^3\,U_{3-j}^2\,z \\
& + 3\,\alpha_2\,U_j\,z\,\alpha_1\,DTT\,U_{3-j} - 3\,\alpha_2\,U_j^2\,z\,\alpha_1\,DTT + DTT^{3/2}\,U_{3-j} \\
& - 2\,DTT^{3/2}\,U_j + 2\,\alpha_2^3\,U_j^3\,U_{3-j}^4 - 4\,\alpha_2^3\,U_j^4\,U_{3-j}^3 + 2\,\alpha_2^3\,U_j^5 \\
& U_{3-j}^2 - 6\,U_j^3\,\alpha_1^3\,U_{3-j}^4 + 6\,U_j^4\,\alpha_1^3\,U_{3-j}^3 + 2\,U_j^2\,\alpha_1^3\,U_{3-j}^5 - 2 \\
& U_j^5\,\alpha_1^3\,U_{3-j}^2 + 3\,\alpha_2^2\,U_j^4\,U_{3-j}\sqrt{DTT} - 3\,U_j^2\,\alpha_1^2\,U_{3-j}^3\sqrt{DTT} \\
& + 6\,U_j^3\,\alpha_1^2\,U_{3-j}^2\sqrt{DTT} - 3\,U_j^4\,\alpha_1^2\,U_{3-j}\sqrt{DTT} + 3\,\alpha_2^2\,U_j^2 \\
& U_{3-j}^3\sqrt{DTT} - 6\,\alpha_2^2\,U_j^3\,U_{3-j}^2\sqrt{DTT}\Big)
\end{aligned} \tag{34}$$

In this case the resulting control value resulting from (18) is the most complicated one

$$\begin{aligned}
u = {} & \frac{1}{3}\Big(-6\,U_{3-j}^2\,\alpha_3\sqrt{DTT}\,x\,\alpha_2^3\,\alpha_1^3 - 4\,U_{3-j}^2\,\alpha_3\,\alpha_2^4\,z^2\,\alpha_1^4\,y + 12 \\
& U_{3-j}^2\,U_j\,\alpha_1^4\,\alpha_2^4\,z\,y + 6\,U_{3-j}^3\,\alpha_3\,U_j\,\alpha_1^4\,y\,\alpha_2^2 + 2\,U_{3-j}^3\,\alpha_3\,U_j\,\alpha_2^4\,y \\
& \alpha_1^2 - U_{3-j}^2\,\alpha_3\,\alpha_2^4\,U_j\,z^2\,\alpha_1^2 + 2\,U_{3-j}^2\,\alpha_3\,U_j^2\,\alpha_2^4\,y\,\alpha_1^2 - 8 \\
& U_{3-j}^2\,\alpha_3\,U_j\,y^2\,\alpha_1^4\,\alpha_2^4 - 3\,U_{3-j}^2\,\alpha_3\,U_j\,\alpha_2^2\,z^2\,\alpha_1^4 - 2\,U_{3-j}^2\,\alpha_3\,U_j^2 \\
& \alpha_1^4\,y\,\alpha_2^2 + U_{3-j}\,\alpha_3\,U_j^2\,\alpha_2^2\,z^2\,\alpha_1^4 - U_{3-j}\,\alpha_3\,U_j^2\,\alpha_2^4\,z^2\,\alpha_1^2 + \alpha_3 \\
& \alpha_1^4\,U_{3-j}^5 - 2\,\alpha_3\sqrt{DTT}\,\alpha_2^3\,z^3\,\alpha_1^3 - 2\,\alpha_3\,U_j\,\alpha_2^4\,z^4\,\alpha_1^4 + 6\,U_{3-j}^2 \\
& \alpha_1^3\,\alpha_2^3\,y\sqrt{DTT} - 6\,U_{3-j}\,\alpha_1^3\,\alpha_2^3\,z^2\sqrt{DTT} - 6\,U_{3-j}^3\,\alpha_1^4\,\alpha_2^4\,z\,y \\
& - 4\,U_{3-j}^4\,\alpha_3\,\alpha_1^4\,y\,\alpha_2^2 + 2\,U_{3-j}^3\,\alpha_3\,\alpha_2^2\,z^2\,\alpha_1^4 + 4\,U_{3-j}^3\,\alpha_3\,y^2\,\alpha_1^4 \\
& \alpha_2^4 + U_{3-j}\,\alpha_3\,\alpha_2^4\,z^4\,\alpha_1^4 - 6\,U_{3-j}^3\,U_j\,\alpha_1^4\,\alpha_2^2\,z + 3\,U_{3-j}^2\,U_j^2\,\alpha_1^4 \\
& \alpha_2^2\,z - 3\,U_{3-j}^2\,U_j^2\,\alpha_1^2\,\alpha_2^4\,z - 6\,U_{3-j}\,U_j\,\alpha_1^4\,\alpha_2^4\,z^3 - 2 \\
& U_{3-j}^2\,\alpha_3\sqrt{DTT}\,U_j\,\alpha_1^3 + 2\,U_{3-j}^2\,\alpha_3\sqrt{DTT}\,U_j\,\alpha_2^3 - \\
& U_{3-j}^4\,\alpha_3\,U_j\,\alpha_1^2\,\alpha_2^2 + 3\,U_{3-j}^3\,\alpha_3\,U_j^2\,\alpha_1^2\,\alpha_2^2 - 2\,U_{3-j}^2\,\alpha_3\,U_j^3\,\alpha_1^2\,\alpha_2^2 \\
& + 2\,U_{3-j}^3\,\alpha_3\sqrt{DTT}\,\alpha_1^3 - U_{3-j}^4\,\alpha_3\,U_j\,\alpha_1^4 - U_{3-j}^3\,\alpha_3\,U_j^2\,\alpha_1^4 + \\
& U_{3-j}^2\,\alpha_3\,U_j^3\,\alpha_1^4 - 2\,U_{3-j}^3\,\alpha_3\,U_j^2\,\alpha_2^4 + U_{3-j}^2\,\alpha_3\,U_j^3\,\alpha_2^4 + 3 \\
& U_{3-j}^4\,\alpha_1^4\,\alpha_2^2\,z + 3\,U_{3-j}^2\,\alpha_1^4\,\alpha_2^4\,z^3 + 6\,U_{3-j}\,\alpha_3\sqrt{DTT}\,\alpha_2^3\,z\,\alpha_1^3\,y \\
& + 8\,U_{3-j}\,\alpha_3\,U_j\,\alpha_2^4\,z^2\,\alpha_1^4\,y\Big) \Big/ \Big(\Big(z\,\alpha_1^2\,U_{3-j}^3 - 2\,U_{3-j}^2\,\alpha_1^2\,z\,y\,\alpha_2^2 \\
& - 2\,U_{3-j}^2\,\alpha_1^2\,U_j\,z + 4\,U_{3-j}\,\alpha_1^2\,\alpha_2^2\,U_j\,z\,y + U_{3-j}\,\alpha_1^2\,\alpha_2^2\,z^3 \\
& + U_{3-j}\,\alpha_1^2\,U_j^2\,z + 2\,U_{3-j}\,y\sqrt{DTT}\,\alpha_2\,\alpha_1 - U_{3-j}\,U_j^2\,\alpha_2^2\,z \\
& - 2\,U_j\,z^3\,\alpha_2^2\,\alpha_1^2 - 2\,z^2\sqrt{DTT}\,\alpha_2\,\alpha_1\Big)\,\alpha_1^2\,\alpha_2^2\Big)
\end{aligned} \tag{35}$$

3.5 Control for Region QT

The last region of the switching surface denoted QT is very similar to the second one denoted TQ. As the name says the combination of parameters q_i and t_i values is opposite to the region TQ. Here the first subsystem $\mathbf{x}_1$ is in the linear case, i.e. its states comply with (9) and the second subsystem $\mathbf{x}_2$ fulfils (10) that means t_2 is nonzero. Therefore we substitute $t_1 = t_2$ and $q_2 = U_{3-j} - q_1$ in (12)

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \Bigg[\Bigg[& q_1\left(\frac{1}{\alpha_1^3} - \frac{t_2}{\alpha_1^2} + \frac{1}{2}\,\frac{t_2^2}{\alpha_1}\right) - \frac{1}{6}\,U_j\,t_2^3 + \big(U_{3-j} \\
& - q_1\big)\left(\frac{1}{\alpha_2^3} - \frac{t_2}{\alpha_2^2} + \frac{1}{2}\,\frac{t_2^2}{\alpha_2}\right) - \left(\frac{1}{6}\,U_{3-j} - \frac{1}{6}\,U_j\right)t_2^3\Bigg], \\
& \Bigg[q_1\left(\frac{1}{\alpha_1^2} - \frac{t_2}{\alpha_1}\right) + \frac{1}{2}\,U_j\,t_2^2 + \left(U_{3-j} - q_1\right)\left(\frac{1}{\alpha_2^2} - \frac{t_2}{\alpha_2}\right) \\
& + \left(\frac{1}{2}\,U_{3-j} - \frac{1}{2}\,U_j\right)t_2^2\Bigg], \\
& \left[\frac{q_1}{\alpha_1} - U_j\,t_2 + \frac{U_{3-j} - q_1}{\alpha_2} - \left(U_{3-j} - U_j\right)t_2\right]\Bigg]
\end{aligned} \tag{36}$$

First we solve parameters q_1 and t_2 from the last two equations of the set (36). From the solution of the quadratic equation we choose that one that gives the positive solution of t_2. After introducing discriminant DQT

$$DQT = \alpha_2^2\,U_{3-j}^2 + z^2\,\alpha_1^2\,\alpha_2^2 + \alpha_1^2\,U_{3-j}^2 - 2\,y\,\alpha_1^2\,\alpha_2^2\,U_{3-j} \tag{37}$$

we get

$$q_1 = \frac{-\alpha_2\,U_{3-j} + sign\left(U_j\right)\sqrt{DQT}}{\alpha_1 - \alpha_2} \tag{38}$$

$$t_2 = -\frac{-\alpha_1\,U_{3-j} + z\,\alpha_1\,\alpha_2 - \alpha_2\,U_{3-j} + sign\left(U_j\right)\sqrt{DQT}}{\alpha_1\,\alpha_2\,U_{3-j}} \tag{39}$$

To express the points of the region QT we substitute (38) and (39) into the first equation of the (36)

$$\begin{aligned}
x = {} & \frac{1}{6}\,\frac{1}{U_{3-j}^2\,\alpha_1^3\,\alpha_2^3}\Big(3\,\alpha_1^3\,U_{3-j}^2\,z\,\alpha_2 + z^3\,\alpha_1^3\,\alpha_2^3 + 2\,\alpha_1^3\,U_{3-j}^3 \\
& + 3\,z\,\alpha_1\,\alpha_2^3\,U_{3-j}^2 - 3\,sign\left(U_j\right)^2 DQT\,z\,\alpha_1\,\alpha_2 + 2\,\alpha_2^3\,U_{3-j}^3 \\
& - 2\,sign\left(U_j\right)^3 DQT^{3/2}\Big)
\end{aligned} \tag{39}$$

The control value results from (18)

$$\begin{aligned}
u = {} & \frac{1}{3}\Big(-3\,y\,\alpha_1^3\,\alpha_2^3\,U_{3-j}^2 + 3\,\alpha_1^3\,\alpha_2^3\,U_{3-j}\,z^2 + 3\,\alpha_1^2 \\
& \alpha_2^2\,U_{3-j}\,z\,sign\left(U_j\right)\sqrt{DQT} + 3\,\alpha_3\,x\,\alpha_1^3\,\alpha_2^3\,U_{3-j}^2 + \alpha_3\,z^3\,\alpha_1^3\,\alpha_2^3 \\
& - 3\,\alpha_3\,\alpha_1^3\,z\,\alpha_2^3\,y\,U_{3-j} - \alpha_3\,\alpha_1^3\,U_{3-j}^3 - \alpha_3\,\alpha_2^3\,U_{3-j}^3 \\
& + \alpha_3 sign\left(U_j\right) DQT^{3/2}\Big) \Big/ \Big(\alpha_1^2\,\alpha_2^2\left(z^2\,\alpha_1\,\alpha_2 - \alpha_1\,\alpha_2\,y\,U_{3-j}\right. \\
& \left. + sign\left(U_j\right)\sqrt{DQT}\,z\right)\Big)
\end{aligned} \tag{40}$$

3.6 Control Algorithm

The control algorithm consists in the localization of the initial state to one of the above mentioned regions and then of the control value calculation. But before we have to specify the parameter j. Then we can calculate the parameters q_i. According their values we can find the region to which the initial point belongs and finally evaluate the control value.

START

1. Evaluate q_1 according (20) and q_2 according (21)
2. IF q_1 fulfils (9) AND q_2 fulfils (9) THEN calculate u according (23) – *Region QQ*

3. IF q_1 fulfils (9) AND q_2 NOT fulfils (9) THEN calculate $j = \frac{3 - sign(q_2)}{2}$ AND GOTO 8
4. Calculate $j = \frac{3 + sign(q_1)}{2}$, $q_1 = U_j$ and q_2 according (26)
5. IF q_2 fulfils (9) THEN calculate u according (29) – *Region TQ*
6. IF $q_2 U_j < 0$ THEN calculate u according (35) – *1st part of the region TT*
7. Calculate $j = \frac{3 - sign(q_1)}{2}$
8. Evaluate q_1 according (38)
9. IF q_1 fulfils (9) THEN calculate u according (40) - *Region QT*
10. Evaluate u according (35) – *2nd part of the region TT*

END

It is important to notice that at the end the control value computed according this algorithm must be limited by (2).

4. EVALUATION OF DESIGNED CONTROLLER

To show the performance of the designed controller we have carried out several simulations that differ from the starting point, parameters of the controller, and constraints. In the Fig. 3 one can see the time responses from the initial state $\mathbf{x} = [200 \quad 0 \quad 0]^t$ under nonsymmetrical control value constraints $u = \langle -1 \quad 2 \rangle$.

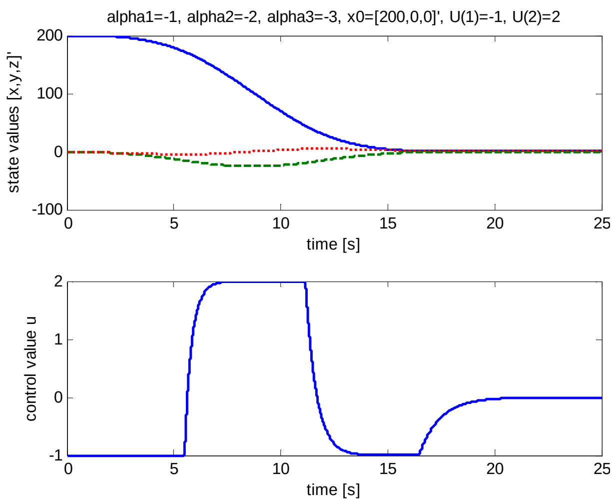

Fig. 3. Time responses of state and control variables. Sub-optimal controller with nonsymmetrical constraints.

All three pulses of the time optimal control can be mentioned but the control value switches from one limit value to the other one smoothly. The change rate is given by the choice of parameters α_i. In this case the values of α_i were $\alpha_1 = -1$, $\alpha_2 = -2$, $\alpha_3 = -3$.

The comparison of the time sub-optimal control with the optimal one is shown in the Fig. 4. This time the starting point was $\mathbf{x} = [15.7916 \quad -4.25 \quad 1]^t$. The other parameters were $\alpha_1 = -1.5$, $\alpha_2 = -3$, $\alpha_3 = -6$ and $u = \langle -1 \quad 1 \rangle$.

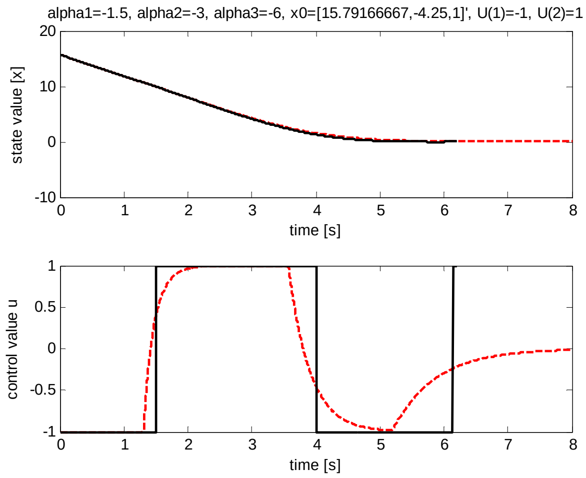

Fig. 4. Comparison of time optimal and sub-optimal control.

There are not big differences in the time responses of the state variable x. But one can see the difference in the behaviour of the control variables. The time sub-optimal control variable uses limits for a shorter period because it needs a certain time to switch to the opposite value. The time optimal control variable switches immediately that can cause problems when the dynamics of a controlled system is not precisely identified. The time sub-optimal controller switches in advance and it finishes later but it is not so sensitive to the uncertain parameters or unmodelled dynamics. By moving the negative values of parameters α_i towards a zero we could get behaviour similar to the linear pole assignment controller. To get the exact linear behaviour we have to higher the control value limits.

Fig. 5 shows the comparison of the time sub-optimal control with the model predictive control (MPC). The example for comparison is taken from (Glattfelder and Schaufelberger, 2003). The parameters of MPC controller were following: prediction horizon $N = 50$, sampling period $T_s = 0.050$, weights of linear quadratic controller $Q = \begin{bmatrix} 11.78 & 0 & 0 \\ 0 & 3.345 & 0 \\ 0 & 0 & 0.3175 \end{bmatrix}$, corresponding bandwidth $\Omega = 3.25$ for $r = 0.01$. Higher Ω-values result in overshooting responses. Parameters of the time sub-optimal controller have been chosen in order to get similar response of control action. They are $\alpha_1 = -3$, $\alpha_2 = -20$, $\alpha_3 = -100$. There are time responses from the initial state $\mathbf{x} = [-1 \quad 0 \quad 0]^t$ under control value constraints $u = \langle -1 \quad 1 \rangle$. One can see that the output response (state value x) is for both controllers almost identical. Very small differences can be shown in the time responses of the control action. This

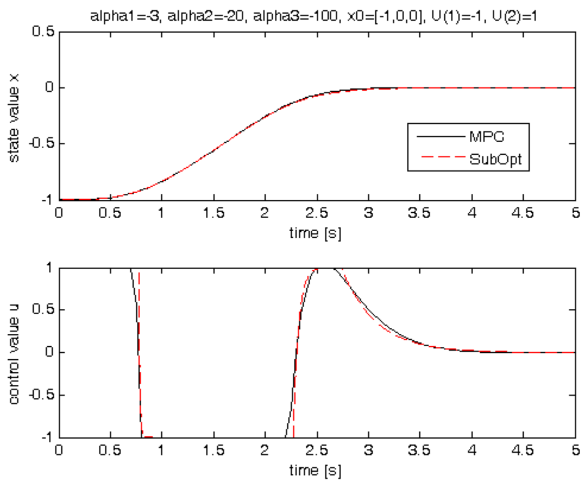

Fig. 5. Comparison of sub-optimal and model predictive control with similar behaviours.

example shows that it is possible to set the parameters of the sub-optimal controller to get equal results with the MPC controller.

Two different responses of the MPC and sub-optimal controllers are shown in the Fig. 6. This time the parameters of both controllers have been decreased. The MPC controller parameters were: prediction horizon $N = 50$, sampling period $T_s = 0.050$, weights of linear quadratic controller $Q = \begin{bmatrix} 2.4413 & 0 & 0 \\ 0 & 1.175 & 0 \\ 0 & 0 & 0.1875 \end{bmatrix}$, corresponding bandwidth $\Omega = 2.5$ for $r = 0.01$. The decrease of parameter values caused that the control action did not reach the control constraint in the last (third) interval of control. One can mention that the switch between the limit values in the first and the second interval of control is almost time optimal. The end of the response (third interval of control) corresponds to the linear controller behaviour.

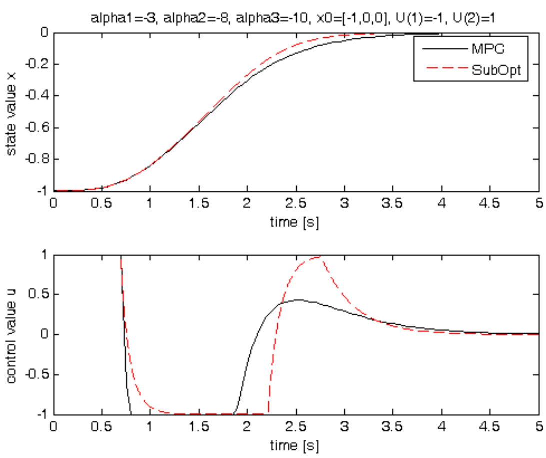

Fig. 6. Comparison of sub-optimal and model predictive control with different behaviours.

Decreasing Ω to the value 1.575 would cause the second interval of control does not reach the limit value and would have linear behaviour too. Thus decreasing the parameter Ω assures the linear character of the control action in the last intervals of control. Choosing the Ω parameter it is not possible to influence the behaviour of switches between the limit control values separately. Opposite to this the sub-optimal controller has three parameters $\alpha_1, \alpha_2, \alpha_3$ that directly influence the exponential behaviour of switches. So it offers to design the control action response more specifically. In this example the values of α_i were $\alpha_1 = -3$, $\alpha_2 = -8$, $\alpha_3 = -10$.

5. CONCLUSIONS

Presented controller design relies on switching surfaces. Because our aim was to decrease the sensitivity of controller we introduced new parameters into the design. In linear cases these parameters are identical with the poles of the closed loop. Of course, the switching surfaces are more complicated by additional parameters. In this paper we derived the solution with explicit mathematical formulas that is fast enough to be used in real time applications. The designed time sub-optimal controller has been compared with pure time optimal controller and model predictive controller.

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