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Optimal Control via Initial State of an Infinite Order Time Delay Hyperbolic System

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Abstract: In this paper, we consider an optimal control problem for a linear infinite order hyperbolic system. One from the initial conditions is given by control function. Sufficient conditions for the existence of a unique solution of such hyperbolic equations with the Dirichlet boundary conditions are presented. The performance functional has the quadratic form. The time horizon T is fixed. Finally, we impose some constraints on the control. Making use of the Lions scheme (Lions (1971)), necessary and sufficient conditions of optimality for the Dirichlet problem with the quadratic performance functional and constrained control are derived.

Keywords: optimal control, infinite order, hyperbolic system, time delay

1. INTRODUCTION

Various optimization problems associated with the optimal control of second order time delay distributed parameter systems have been studied in Wang (1975); Knowles (1978); Kowalewski (1988b 1993ab 1998 2000 2001) respectively.

In Knowles (1978), the time optimal control problems of linear parabolic systems with the Neumann boundary conditions involving constant time delays were considered.

These equations constitute in a linear approximation, a universal mathematical model for many diffusion processes in which time-delayed feedback signals are introduced at the boundary of a system's spatial domain. For example, in the area of plasma control (Wang (1975)), it is of interest to confine a plasma in a given bounded spatial domain Ω by introducing a finite electric potential barrier or a "magnetic mirror" sorrounding Ω. For a collision-dominated plasma, its particle density is describable by a parabolic equation. Due to particle inertia and finiteness of electric potential or the magnetic -mirror field strength, the particle reflection at the domain boundary is not instantaneous. Consequently, the particle flux at the boundary of Ω at any time depends on the flux of particles which escaped earlier and reflected back into Ω at a later time. This leads to the boundary conditions involving time delays.

Using the results of Wang (1975), the existence of a unique solution of such parabolic systems was discussed. A characterization of the optimal control in terms of the adjoint system was given. Consequently, this characterization was used to derive specific properties of the optimal control (bang-bangness, uniqueness, etc.). These results were also extended to certain cases of nonlinear control without convexity and to certain fixed-time problems.

Consequently, in Kowalewski (1988b 1993ab 1998 2000 2001) linear quadratic problems for second order hyperbolic systems with time delays given in the different form (constant time delays, time-varying delays, integral time delays, etc.) were presented.

Moreover, in Lions (1971) and Kowalewski (2004) optimal control problems via initial state for second order hyperbolic systems were investigated.

Such hyperbolic systems constitute in a linear approximation mathematical models of representative convection-reaction processes, e.g. fixed-bed reactors, pressure swing absorption processes, etc.

In particular, in Kowalewski (2010), the optimal control problems via initial condition for infinite order hyperbolic systems were considered. The presented optimal control problem can be generalized onto the case of time delay infinite hyperbolic systems.

For this reason, in the present paper we consider an optimal control problem for a linear time delay infinite order hyperbolic system with constant time delay appearing in the state equation.

We consider a different type of equations, namely, time delay infinite order partial differential equations of hyperbolic type with one from the initial conditions given by control function.

The paper is organized as follows. The existence and uniqueness of solutions for such hyperbolic equations were proved - Lemma 1 and Theorem 2. The optimal control is characterized by the adjoint problem - Lemma 3. The necessary and sufficient conditions of optimality with the quadratic performance functional and constrained control are derived for the Dirichlet problem - Theorem 4.

2. PRELIMINARIES

Let Ω be a bounded open set of R^n with smooth boundary Γ.

We define the infinite order Sobolev space $H^\infty\{a_\alpha, 2\}(\Omega)$ of functions $\Phi(x)$ defined on Ω Dubinskij (1975) and Dubinskij (1976) as follows

$$H^\infty\{a_\alpha,2\}(\Omega) = \\ = \left\{\Phi(x) \in C^\infty(\Omega) : \sum_{|\alpha|=0}^{\infty} a_\alpha \parallel \mathcal{D}^\alpha \Phi \parallel_2^2 < \infty \right\}, \tag{1}$$

where: $C^\infty(\Omega)$ is a space of infinite differentiable functions, $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is a norm in the space $L^2(\Omega)$, and

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \dots (\partial x_n)^{\alpha_n}}\ , \tag{2}$$

where: $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

The space $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ is defined as the formal conjugate space to the space $H^\infty\{a_\alpha, 2\}(\Omega)$, namely:

$$H^{-\infty}\{a_\alpha,2\}(\Omega) = \\ = \left\{\Psi(x) : \Psi(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathcal{D}^\alpha \Psi_\alpha(x) \right\}, \tag{3}$$

where: $\Psi_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \parallel \Psi_\alpha \parallel_2^2 < \infty$.

The duality pairing of the spaces $H^\infty\{a_\alpha, 2\}(\Omega)$ and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ is postulated by the formula

$$\langle \Phi, \Psi \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega \Psi_\alpha(x) \mathcal{D}^\alpha \Phi(x)\ dx, \tag{4}$$

where: $\Phi \in H^\infty\{a_\alpha, 2\}(\Omega)$, $\Psi \in H^{-\infty}\{a_\alpha, 2\}(\Omega)$.

From above, $H^\infty\{a_\alpha, 2\}(\Omega)$ is everywhere dense in $L^2(\Omega)$ with topological inclusions and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ denotes the topological dual space with respect to $L^2(\Omega)$ so we have the following chain:

$$H^\infty\{a_\alpha,2\}(\Omega) \subseteq L^2(\Omega) \subseteq H^{-\infty}\{a_\alpha,2\}(\Omega). \tag{5}$$

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Consider now the distributed-parameter system described by the following infinite order hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} + Ay + y(x, t-h) = u \quad x \in \Omega, t \in (0,T), \tag{6}$$

$$y(x,t') = \Phi_0(x,t') \quad x \in \Omega, t' \in [-h, 0), \tag{7}$$

$$y(x,0) = 0 \qquad x \in \Omega, \tag{8}$$

$$y'(x,0) = v \qquad x \in \Omega, \tag{9}$$

$$y(x,t) = 0 \qquad x \in \Gamma, t \in (0,T), \tag{10}$$

where Ω has the same properties as in the Section 1.

$$y \equiv y(x,t;v), \qquad u \equiv u(x,t), \qquad v \equiv v(x)$$
$$Q = \Omega \times (0,T), \qquad \bar{Q} = \bar{\Omega} \times [0,T],$$
$$Q_0 = \Omega \times [-h, 0), \qquad \Sigma = \Gamma \times (0,T),$$

h is a specified positive number representing a time delay, Φ_0 is an initial function defined on Q_0.

The operator $\frac{\partial^2}{\partial t^2} + A$ in (6) is an infinite order hyperbolic operator and A (Dubinskij (1986)) is given by

$$Ay = \Big(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathcal{D}^{2\alpha} + 1 \Big) y \tag{11}$$

and $\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathcal{D}^{2\alpha}$ is an infinite order elliptic partial differential operator.

The operator A is a mapping of $H^\infty\{a_\alpha, 2\}$ onto $H^{-\infty}\{a_\alpha, 2\}$. For this operator the bilinear form $\Pi(t; y, \varphi) = (Ay, \varphi)_{L^2(\Omega)}$ is coercive on $H^\infty\{a_\alpha, 2\}$ i.e. there exists $\lambda > 0, \lambda \in I\!R$ such that $\Pi(t; y, \varphi) \geq \lambda \|y\|^2_{H^\infty\{a_\alpha,2\}}$. We assume that $\forall\ y, \varphi \in H^\infty\{a_\alpha, 2\}$ the function $t \to \Pi(t; y, \varphi)$ is continuously differentiable in $[0,T]$ and $\Pi(t; y, \varphi) = \Pi(t; \varphi, y)$.

The equations (6) - (10) constitute a Dirichlet problem.

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (6) - (10) for the case where $v \in L^2(\Omega)$.

The existence of a unique solution for the mixwd initial-boundary value problem (6) - (10) on the cylinder Q can be proved using a constructive method, i.e. first solving (6) - (10) on the subcylinder Q_1 and in turn on Q_2, etc., until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

For simplicity, we introduce the notations

$$E_j \stackrel{\triangle}{=} ((j-1)h, jh), \quad Q_j = \Omega \times E_j,\ j = 1, ..., K$$

Using the results of Section 6 of (Lions (1971), p. 314) we can prove the following lemma.

Lemma 1. Let

$$u \in L^2(Q), \tag{12}$$

$$f_j \in L^2(Q_j), \tag{13}$$

where

$$f_j(x,t) = u(x,t) - y_{j-1}(x, t-h),$$
$$y_{j-1}(\cdot, (j-1)h) \in H^\infty\{a_\alpha, 2\}(\Omega), \tag{14}$$
$$y'_{j-1}(\cdot, (j-1)h) \in L^2(\Omega). \tag{15}$$

Then, there exists a unique solution $y_j \in L^2(E_j; H^\infty\{a_\alpha, 2\}(\Omega))$ with $\frac{dy_j}{dt} \in L^2(E_j; L^2(\Omega))$ for the mixed initial-boundary value problem (6), (14) and (15).

Proof. We observe that for $j = 1$ we have $y_{j-1}\big|_{Q_0}(x, t-h) = \Phi_0(x, t-h)$. Then the assumptions

(13), (14) and (15) are fullfilled if we assume that $\Phi_0 \in L^2(-h, 0; H^\infty\{a_\alpha, 2\}(\Omega))$, $\frac{d\Phi_0}{dt} \in L^2(-h, 0; L^2(\Omega))$, $y_0(x, 0) \in H^\infty\{a_\alpha, 2\}(\Omega)$ and $y_0'(x, 0) \in L^2(\Omega)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in L^2(E_1; H^\infty\{a_\alpha, 2\}(\Omega))$ with $\frac{dy_1}{dt} \in L^2(E_1; L^2(\Omega))$.

In order to extend the results to Q_2, we have to to prove that $y_1(\cdot, h) \in H^\infty\{a_\alpha, 2\}(\Omega)$, $y_1'(\cdot, h) \in L^2(\Omega)$ and $f_2 \in L^2(Q_2)$. From the Theorem 3.1 of Lions and Magenes (1972) (Vol.1, p.19) $y_1 \in L^2(E_1; H^\infty\{a_\alpha, 2\}(\Omega))$ jointly with $\frac{dy_1}{dt} \in L^2(E_1; L^2(\Omega))$ imply that the mappings $t \to y_1(\cdot, t)$ and $t \to y_1'(\cdot, t)$ are continuous from $[0, h] \to H^\infty\{a_\alpha, 2\}(\Omega)$ and $[0, h] \to L^2(\Omega)$ respectively. Thus, $y_1(\cdot, h) \in H^\infty\{a_\alpha, 2\}(\Omega)$ and $y_1'(\cdot, h) \in L^2(\Omega)$.

Also it is easy to notice that the assumption (13) follows from the fact that $y_1 \in L^2(E_1; H^\infty\{a_\alpha, 2\}(\Omega))$ and $u \in L^2(Q)$. Thus, there exists a unique solution $y_2 \in L^2(E_2; H^\infty\{a_\alpha, 2\}(\Omega))$ with $\frac{dy_2}{dt} \in L^2(E_2; L^2(\Omega))$.□

The foregoing result is now summarized for $j = 1, ..., K$.

Theorem 2. Let $y(x, 0)$, $y'(x, 0), \Phi_0$ and u be given with $y(\cdot, 0) \in H^\infty\{a_\alpha, 2\}(\Omega)$, $y'(\cdot, 0) \in L^2(\Omega)$, $\Phi_0 \in L^2(-h, 0; H^\infty\{a_\alpha, 2\})$, $\frac{d\Phi_0}{dt} \in L^2(-h, 0; L^2(\Omega))$ and $u \in L^2(Q)$. Then, there exists a unique solution $y \in L^2(0, T; H^\infty\{a_\alpha, 2\}(\Omega))$ with $\frac{dy}{dt} \in L^2(0, T; L^2(\Omega))$ for the mixed initial-boundary value problem (6) - (10).

4. PROBLEM FORMULATION. OPTIMIZATION THEOREM.

We shall now formulate the optimal control problem for the Dirichlet problem. Let us denote by $U = L^2(\Omega)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \int_\Omega |\ y(x, T; v) - z_d\ |^2\ dx + \lambda_2 \int_\Omega (Nv)v\ dx, \tag{16}$$

where: $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $L^2(\Omega)$; N is a positive linear operator on $L^2(\Omega)$ into $L^2(\Omega)$.

Finally, we assume the following constraint on controls $v \in U_{ad}$, where

$$U_{ad} \text{ is a closed, convex subset of } U. \tag{17}$$

Let $y(x, t; v)$ denote the solution of the mixed initial-boundary value problem (6) - (10) at (x, t) corresponding to a given control $v \in U_{ad}$. We note from the Theorem 2 that for any $v \in U_{ad}$ the performance functional (16) is well-defined since $y(x, T; v) \in H^\infty\{a_\alpha, 2\}(\Omega) \in L^2(\Omega)$. The solving of the formulated optimal control problem is equivalent to seeking a $v_0 \in U_{ad}$ such that $I(v_0) \leq I(v)\ \forall v \in U_{ad}$.

Then from the Theorem 1.3 (Lions (1971), p. 10) it follows that for $\lambda_2 > 0$ a unique optimal control v_0 exists; moreover, v_0 is characterized by the following condition

$$I'(v_0) \cdot (v - v_0) \geq 0 \qquad \forall v \in U_{ad}. \tag{18}$$

Using the form of the performance functional (16) we can express (18) in the following form

$$\lambda_1 \int_\Omega (y(x, T; v_0) - z_d)(y(x, T; v) - y(x, T; v_0))dx + \lambda_2 \int_\Omega (Nv_0)(v - v_0)dx \geq 0 \quad \forall v \in U_{ad}. \tag{19}$$

To simplify (19), we introduce the adjoint equation and for every $v \in U_{ad}$, we define the adjoint variable $p = p(v) = p(x, t; v)$ as the solution of the equation

$$\frac{\partial^2 p(v)}{\partial t^2} + Ap(v) + p(x, t + h; v) = 0 \qquad x \in \Omega,\ t \in (0, T - h), \tag{20}$$

$$\frac{\partial^2 p(v)}{\partial t^2} + Ap(v) = 0 \quad x \in \Omega, t \in (T - h, T), \tag{21}$$

$$p(x, T; v) = 0 \qquad x \in \Omega, \tag{22}$$

$$p'(x, T; v) = -\lambda_1(y(x, T; v) - z_d) \quad x \in \Omega, \tag{23}$$

$$p(x, t) = 0 \quad x \in \Gamma,\ t \in (0, T). \tag{24}$$

The existence of a unique solution for the problem (20)-(24) on the cylinder Q can be proved using a constructive method. It is easy to notice that for given z_d and v, problem (20)-(24) can be solved backwards in time starting from $t = T$, i.e., first, solving (20)-(24) on the subcylinder Q_K and in turn on Q_{K-1}, etc. until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 2 (with an obvious change of variables) to problem (20)-(24).

Lemma 3. Let the hypothesis of Theorem 2 be satisfied. Then, for given $z_d \in L^2(\Omega)$ and any $v \in L^2(\Omega)$, there exists a unique solution such that $p(v) \in L^2(0, T; H^\infty\{a_\alpha, 2\}(\Omega))$ and $\frac{\partial p(v)}{\partial t} \in L^2(0, T; L^2(\Omega))$ for the problem (20)-(24).

We simplify (19) using the adjoint equation (20)-(24). For this purpose setting $v = v_0$ in (20)-(24), multiplying both sides of (20)-(21) by $(y(v) - y(v_0))$ and then integrating over $\Omega \times (0, T - h)$ and $\Omega \times (T - h, T)$ respectively and then adding both sides of (20), (21) we get

$$\int_Q \left(\frac{\partial^2 p(v_0)}{\partial t^2} + Ap(v_0)\right)(y(v) - y(v_0))\ dxdt + \int_0^{T-h} \int_\Omega p(x, t + h; v_0)(y(x, t; v) - y(x, t; v_0))dxdt = \int_\Omega p'(x, T, v_0)[y(x, T; v) - y(x, T; v_0)]dx$$

$$
\begin{aligned}
&-\int_\Omega p'(x,0,v_0)[y(x,0;v)-y(x,0;v_0)]dx \\
&-\int_\Omega p(x,T,v_0)[y'(x,T;v)-y'(x,T;v_0)]dx \\
&+\int_\Omega p(x,0,v_0)[y'(x,0;v)-y'(x,0;v_0)]dx \\
&+\int_Q p(v_0)\frac{\partial^2}{\partial t^2}(y(v)-y(v_0))\ dxdt \\
&+\int_Q Ap(v_0)(y(v)-y(v_0))\ dxdt \\
&+\int_0^{T-h}\int_\Omega p(x,t+h;v_0)\cdot \\
&\qquad\cdot(y(x,t;v)-y(x,t;v_0))dxdt = 0 \qquad (25)
\end{aligned}
$$

and so, by (22) and (23),

$$
\begin{aligned}
&\lambda_1\int_\Omega (y(x,T,v)-z_d)[y(x,T;v)-y(x,T;v_0)]dx \\
&=\int_\Omega p(x,0,v_0)[y'(x,0;v)-y'(x,0;v_0)]dx \\
&+\int_Q p(v_0)\frac{\partial^2}{\partial t^2}(y(v)-y(v_0))\ dxdt \\
&+\int_Q Ap(v_0)(y(v)-y(v_0))dxdt \\
&+\int_0^{T-h}\int_\Omega p(x,t+h;v_0)\cdot \\
&\qquad\cdot(y(x,t;v)-y(x,t;v_0))dxdt \qquad (26)
\end{aligned}
$$

Using the equation (6), the second integral on the right-hand side of (26) can be rewritten as

$$
\begin{aligned}
&\int_Q p(v_0)\frac{\partial^2}{\partial t^2}(y(v)-y(v_0))dxdt \\
&=-\int_Q p(v_0)A(y(v)-y(v_0))dxdt \\
&-\int_0^{T}\int_\Omega p(x,t;v_0)\cdot \\
&\qquad\cdot(y(x,t-h;v)-y(x,t-h;v_0))dxdt \\
&=-\int_Q p(v_0)A(y(v)-y(v_0))dxdt \\
&-\int_{-h}^{T-h}\int_\Omega p(x,t'+h;v_0)\cdot \\
&\qquad\cdot(y(x,t';v)-y(x,t';v_0))dxdt' \qquad (27)
\end{aligned}
$$

Substituting (27) into (26) we obtain

$$
\begin{aligned}
&\lambda_1\int_\Omega (y(x,T;v_0)-z_d)(y(x,T;v)-y(x,T;v_0))dx \\
&=\int_\Omega p(x,0,v_0)[y'(x,0;v)-y'(x,0;v_0)]dx \\
&-\int_Q p(v_0)A(y(v)-y(v_0))dxdt \\
&-\int_{-h}^{0}\int_\Omega p(x,t+h;v_0)(y(x,t;v)-y(x,t;v_0))dxdt \\
&-\int_0^{T-h}\int_\Omega p(x,t+h;v_0)(y(x,t;v)-y(x,t;v_0))dxdt \\
&+\int_Q p(v_0)A(y(v)-y(v_0))dxdt \\
&+\int_0^{T-h}\int_\Omega p(x,t+h;v_0)\cdot \\
&\qquad\cdot(y(x,t;v)-y(x,t;v_0))dxdt \qquad (28)
\end{aligned}
$$

Afterwards using the formulae $y'(x,0;v) = v$ and $y'(x,0;v_0)=v_0$ in (28) we get

$$
\begin{aligned}
&\lambda_1\int_\Omega (y(x,T;v_0)-z_d)(y(x,T;v)-y(x,T;v_0))dx \\
&\qquad=\int_\Omega p(x,0;v_0)(v-v_0)dx \qquad (29)
\end{aligned}
$$

Substituting (29) into (19) we obtain

$$
\int_\Omega (p(x,0;v_0)+\lambda_2 Nv_0)(v-v_0)dx \geq 0 \qquad \forall v\in U_{ad} \qquad (30)
$$

Theorem 4. For the problem (6)-(10) with the performance functional (16) with $z_d \in L^2(\Omega)$ and $\lambda_2 > 0$ and with constraints on controls (17), there exists a unique optimal control v_0 which satisfies the maximum condition (30).

We must notice that the conditions of optimality derived above (Theorem 4) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on controls). This results from the following: the determining of the function $p(v_0)$ in the maximum condition from the adjoint equation is possible if and only if we know y_0 which corresponds to the control v_0. These mutual connections make the practical use of the derived optimization formulas difficult. Therefore we resign from the exact determining of the optimal control and we use approximation methods.

In the case of the performance functional (16) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one (Kowalewski (1988a)) which can be solved by the use of the well-known algorithms, e.g. Gilbert's in Kowalewski (1988a).

5. CONCLUSIONS

The results presented in the paper can be treated as a generalization of the results obtained in Kowalewski (2010) onto the case of infinite order time delay hyperbolic systems with one of the initial conditions given by control function.

In this paper we have considered optimal control problem for such hyperbolic systems with the Dirichlet boundary conditions.

We can also consider similar optimal control problems for time delay infinite order hyperbolic systems with Neumann boundary conditions.

Finally we can consider optimal control problem for infinite order hyperbolic systems with two initial conditions given by control functions.

The ideas mentioned above will be developed in forthcoming papers.

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