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MATLAB TOOLBOX FOR PWA IDENTIFICATION OF NONLINEAR SYSTEMS

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Abstract: This paper is dedicated to issue of approximation of nonlinear functions and nonlinear dynamical systems by Piecewise Affine (PWA) linear model. The article presents new identification Matlab toolbox for modelling and simulation of nonlinear systems. Functions of the toolbox together with GUI application simplified and accelerates identification of so called PWA OAF model. Identification of nonlinear systems is based on novel method of PWA modelling by generalized Fourier series. The approach provides identification of nonlinear functions of an arbitrary number of variables and identification of nonlinear dynamical systems in ARX model structure fashion from input-output data.

Keywords: PWA systems, Generalized Fourier series, Matlab toolbox, Chebyshev polynomial, PWA identification.

1. INTRODUCTION

In the recent research many methods were developed for modelling of hybrid systems and general nonlinear functions at all (Roll et al., 2004; Ferrari-Trecate, 2005; Julian et al., 1999). Many model structures were developed for hybrid systems and nonlinear systems. Much attention is dedicated to system modeling in MLD (Mixed Integer Dynamical) form (Bemporad and Morari, 1999) and PWA (Piecewise Affine). In (Bemporad et al., 2000), the formal equivalence between MLD systems and PWA systems is established and also effective algorithms were developed for transformation from one model structure to another (Villa et al., 2004; Bemporad, 2002). In (Heemels et al., 2001ab), the equivalence between the following five classes of hybrid systems is, under certain conditions, established: MLD systems, Linear Complementarity (LC) systems, Extended Linear Complementarity (ELC) systems, PWA systems and Max-Min-Plus-Scaling (MMPS) systems. The important result of these equivalences is that derived theoretical properties and tools can easily be transferred from one class to another.

In this paper we present an effective tool for modeling of nonlinear systems by PWA using novel approach based on generalized Fourier series (Kozak and Stevek, 2010). This approach belongs to black-box identification methods of general nonlinear models (Sjöberg et al., 1995).

We use methodology of generalized Fourier series with orthogonal polynomials. In (Leondes, 1997), orthogonal polynomials were used as activation functions for special case of neural network with one hidden layer - Orthogonal Activation Function based Neural Network (OAF NN). For this type of neural network online and off-line training algorithm has been defined with fast convergence properties. After simple modification of OAF NN it is possible to use this technique for PWA ap-

proximation of a common nonlinear system.
The paper is divided in six sections. First, we formulate the identification and linearization problem of nonlinear function. Next, we present modeling of nonlinear process by OAF NN, topology of the fourier series (PWA OAF NN) and network transformation to state space PWA form. In Section 3 PWA OAF identification toolbox is presented on three case studies. In Section 3.3 is identified nonlinear dynamical system from input-output data and designed explicit mpc control law.

2. PROBLEM FORMULATION

PWA linear approximation of hybrid systems depends on defining guardlines of the PWA mappping. If guardlines are known, the problem of identifying PWA systems can easily be solved using standard techniques for linear systems (Roll et al., 2004). The method based on finding mapping guardlines is suitable for linear system with nonlinear discrete parts like switches which changes system behavior in step. Other methods a priori assume that the system dynamics is continuous (Ferrari-Trecate, 2005). Both mentioned approaches use for identification clustering-based algorithms.
As will be pointed out, nonlinear identification techniques can be used under specific conditions in order to obtain linear PWA model. Many neural network based identification techniques use nonlinear neuron functions of one variable which are easier linearizable than whole model of many variables. The key idea is based on linearization of nonlinear neural network functions of single variable. Similarly as Taylor series, it is possible to define any nonlinear function as a series of nonlinear functions. This approach leads to generalized Fourier polynomial series. Generalized Fourier series is based on a set of one-dimensional orthonormal functions $\phi_i^{(N)}$ defined as

$$\int_{x_1}^{x_2} \phi_i^{(N)}(x)\phi_j^{(N)}(x) = \delta_{ij} \tag{1}$$

where δ_{ij} is the Kronecker delta function and $[x_1, x_2]$ is the domain of interest. Several examples of orthonormal functions are the normalized Fourier (harmonic) functions, Legendre polynomials, Chebyshev polynomials and Laguerre polynomials (Leondes, 1997). In this paper only Chebyshev polynomials will be discussed.
Orthogonal Activation Function based Neural Network (OAF NN) is employed in the task of nonlinear approximation. PWA approximation of every used orthonormal polynomial creates Piecewise Affine Orthogonal Activation Function based Neural Network (PWA OAF NN).

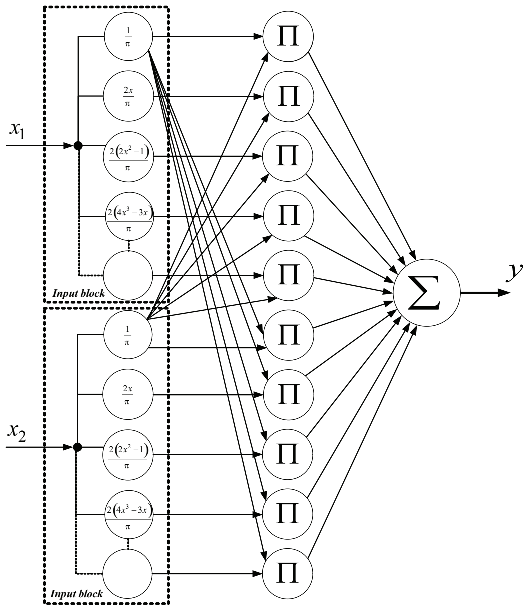


Fig. 1. Adjusted OAF NN structure

2.1 *Chebyshev polynomial*

The Chebyshev polynomials of the first kind can be defined by the trigonometric identity

$$T_n(x) = \cos(n \arccos(x)) \tag{2}$$

with norm defined as follows

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} (T_n(x))^2 dx = \begin{cases} \pi & n = 0 \\ \pi/2 & n \neq 1 \end{cases} \tag{3}$$

Recursive generating formula for Chebyshev polynomials:

$$T_0(x) = 1, \tag{4}$$
$$T_1(x) = x, \tag{5}$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{6}$$
$$T_n(x) = U_{n+1}(x) - U_{n-1}(x). \tag{7}$$

where U_n is the Chebyshev polynomial of the second kind generated by the recursive formula:

$$U_0(x) = 1, \tag{8}$$
$$U_1(x) = 2x, \tag{9}$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \tag{10}$$

The first few Chebyshev polynomials of the first kind are

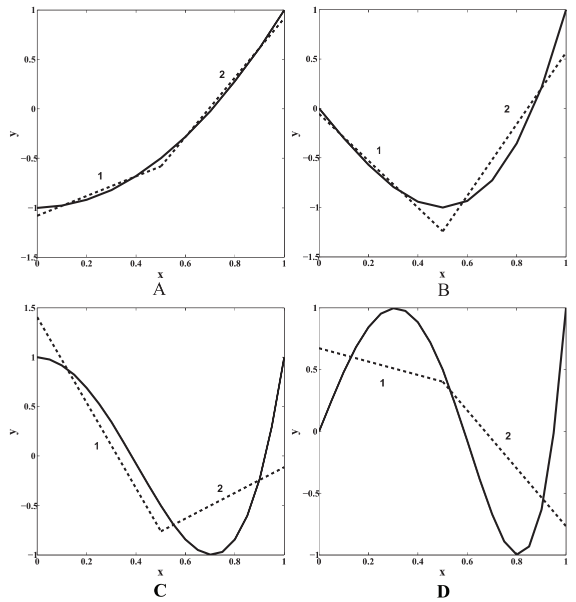


Fig. 2. PWA approximation of T2, T3, T4, T5 Chebyshev polynomials

$$T_0(x) = 1, \tag{11}$$
$$T_1(x) = x, \tag{12}$$
$$T_2(x) = 2x^2 - 1 \tag{13}$$
$$T_3(x) = 4x^3 - 3x \tag{14}$$
$$T_4(x) = 8x^4 - 8x^2 + 1. \tag{15}$$

The first few Chebyshev polynomials of the second kind are

$$U_0(x) = 1, \tag{16}$$
$$U_1(x) = 2x, \tag{17}$$
$$U_2(x) = 4x^2 - 1 \tag{18}$$
$$U_3(x) = 8x^3 - 4x \tag{19}$$
$$U_4(x) = 16x^4 - 12x^2 + 1. \tag{20}$$

2.2 *OAF NN topology*

It is possible to define Generalized Fourier series with orthogonal polynomilas by neural network with one hidden layer. In this work we use a Matlab function framework for orthogonal activation function based neural networks which is part of the toolbox. Aftr slight revision it is possible to use this methodology for modeling the fourier series. Example of the network for modeling function of two variables is depicted in Fig. 1.

If we consider general structure of the network in ARX fashion with na, nb, and nk parameters we get network output equation:

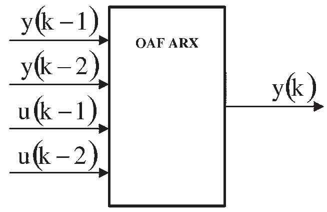


Fig. 3. OAF ARX model for na=2, nb=2, nk=0 or nk=1

$$\begin{aligned}
y &= w_1 \frac{1}{\pi}^p + \dots \\
&\frac{2}{\pi^p}(w_2 T_1(y(k|1)) + \dots + w_n T_{n-1}(y(k|1)))+ \\
&\vdots \\
&\frac{2}{\pi^p}(w_{i_1} T_1(y(k|na)) + \dots + w_{i_2} T_{n-1}(y(k|na)))+ \\
&\frac{2}{\pi^p}(w_{i_3} T_1(u(k|nk)) + \dots + w_{i_4} T_{n-1}(u(k|nk)))+ \\
&\vdots \\
&\frac{2}{\pi^p}(w_{i_5} T_1(u(k|i_7)) + \dots + w_{i_6} T_{n-1}(u(k|i_7))).
\end{aligned}$$

$$\begin{aligned}
i_1 &= (na-1)(n-1)+2 \\
i_2 &= na(n-1)+1 \\
i_3 &= nb(n-1)+2 \\
i_4 &= (nb+1)(n-1)+1 \\
i_5 &= (nb+na-1)(n-1)+2; \\
i_6 &= (nb+na)(n-1)+1; \\
i_7 &= nk+nb-1 \\
p &= na+nb
\end{aligned} \tag{21}$$

where $y(k|na)$ denotes $y(k-na)$ and similarly $u(k|nk) \equiv u(k-nk)$. Every Chebyshev polynomial is aproximated by set of lines (Fig. 2)

$$T(x) \approx a_i x + b_i \quad \text{for} \quad i = \{1, 2, \dots, n_{div}\} \tag{22}$$

Then output equation becomes difference equation.

A convenient feature of all Chebyshev polynomial is their symmetry. All polynomials of even order are symmetrical by vertical axis and all polynomial of odd order are symmetrical by origin. These properties allow decreasing number of linearization points to half while keeping precision. To get the lowest number of shift cases of generated PWA model we linearized the polynomials in the same points, Fig. 2. The term 'linearization point' denotes the interval division point where the PWA function breaks.

2.3 *Transformation to state space PWA form*

Accuracy of the approximation of nonlinear system is significantly increased when the function is linearized around multiple distinct linearization points. State space PWA structure describes behavior of nonlinear dynamical systems in multiple linearization points.

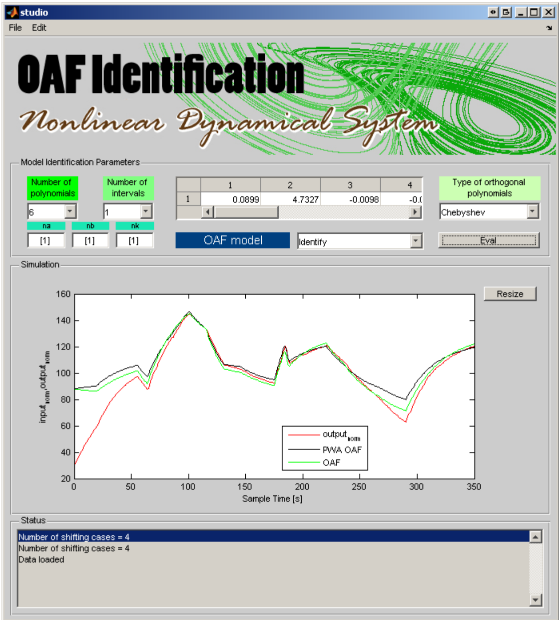


Fig. 4. PWA OAF ID studio

$$
\begin{aligned}
x(k+1) &= A_i x(k) + B_i u(k) + f_i \\
y(k) &= C_i x(k) + D_i u(k) + g_i
\end{aligned} \tag{23a}
$$

$$
IF \begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{D}_i, \quad i = 1, \ldots, n_L \tag{23b}
$$

Every dynamic i is active in polyhedral partition (23b) which can be expressed by inequality

$$
guardX_i x(k) + guardU_i u(k) \le guardC_i \tag{24}
$$

Difference equation (21) can be easily transformed to state space form. In Matlab difference equation can be expressed by discrete transfer function. It is possible to use transformation function `tf2ss`. But this policy doesn't lead to desired state space PWA form. Desired state space form has to keep all outputs of difference equation (21) in state vector. So we can correctly define guardline inequality (24).

Here we present transformation example for system with parameters **na=2, nb=2, nk=0 or nk=1**, Fig.3 . Difference equation:

$$
\begin{aligned}
y(k) = {} & c^{(i)} + c^{(i)}_{y_1} y(k-1) + c^{(i)}_{y_2} y(k-2) + \\
& c^{(i)}_{u_1} u(k-1) + c^{(i)}_{u-2} u(k-2)
\end{aligned} \tag{25}
$$

In PWA form guidelines are defined for $x1 = u(k-2), x2 = y(k-2), x3 = y(k-1)$ and $u = u(k-1)$ PWA state space model:

$$
x(k+1) = A_i x(k) + B_i u(k) + f_i \tag{26a}
$$
$$
y(k) = C_i x(k) + D_i(k) + g_i \tag{26b}
$$
$$
A_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ c^{(i)}_{u_2} & c^{(i)}_{y_2} & c^{(i)}_{y_1} \end{bmatrix} \tag{26c}
$$
$$
B_i = \begin{bmatrix} 1 \\ 0 \\ c^{(i)}_{u_1} \end{bmatrix} \tag{26d}
$$
$$
C_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{26e}
$$
$$
D_i = 0 \tag{26f}
$$
$$
f_i = \begin{bmatrix} 0 \\ 0 \\ c^{(i)} \end{bmatrix} \tag{26g}
$$
$$
g_i = 0; \tag{26h}
$$
$$
x \in < 3 \times 1 > \tag{26i}
$$

3. PWA OAF IDENTIFICATION TOOLBOX

PWA identification problem has garnered great interest in the research community. In Matlab enviroment several toolboxes were developed for identification hybrid and nonlinear systems (Roll et al., 2004; Ferrari-Trecate, 2005; Julian et al., 1999). The main aim of the PWA OAF Identification Toolbox (PWA OAF IT) is to provide efficient tools for analysis, identification and simulation of PWA OAF model. In following section we present toolbox functionality on several identification examples.

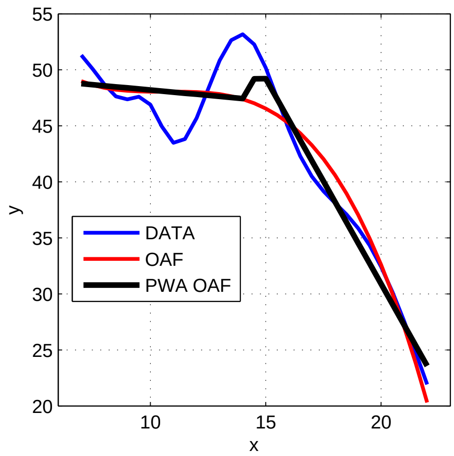
In PWA OAF IT the model is represented by the following fields of the model structure:

`model.na` - Number of past output terms
`model.nb` - Number of past input terms
`model.nk` - Delay from input to the output
`model.npoly` - Number of Chebys. polynomials
`model.ndiv` - Division of {0,1} interval
`model.Fi` - Connection matrix of network
`model.w` - Network parameters
`model.type` - Type of polynomials 'Chebys'
`model.const` - Constant in difference equation
`model.yconst` - Y-cons in difference equation
`model.uconst` - U-const in difference equation
`model.sysStruct` - PWA state space struct
`model.ynorm` - Normalized output data
`model.unorm` - Normalized input data
`model.u` - Input data
`model.y` - Output data
`model.ypar` - Normalization param. of output
`model.upar` - Normalization param. of input

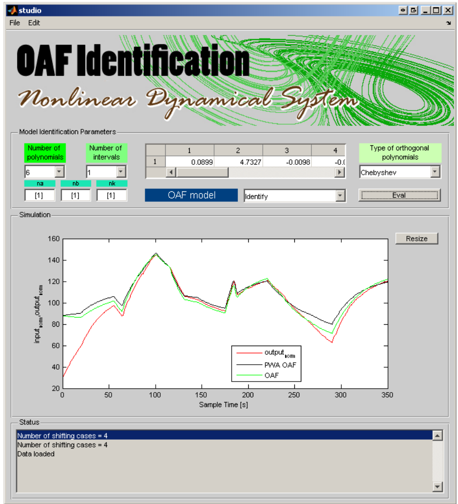
So far PWA OAF ID supports only MISO systems. In order to obtain identified model, call

```
>>model = pwaoafid(y,u,modelstruct,param)
```

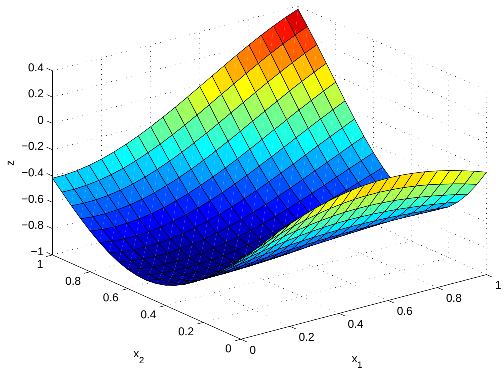
Input arguments are in standard notation well



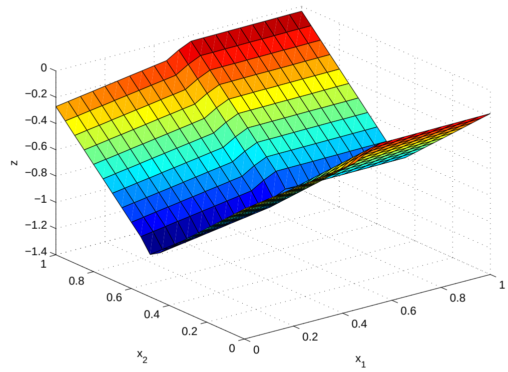
(a) 2-D function example



(b) PWA-OAF Identification Studio



(a) 3-D function data



(b) PWA approximation of 3-D function

Fig. 5. 3D function

known from PWAID toolbox. For more information type

```
>>help oafpwaid
```

For using gui application Fig. 4, call

```
>> oafpwaid_studio
```

3.1 *Identification of 2-D function*

2-D function is defined by formula:

$$y = a_1 e^{-((x-b_1)/c_1)^2} + a_2 e^{-((x-b_2)/c_2)^2} + a_3 e^{-((x-b_3)/c_3)^2} + a_4 e^{-((x-b_4)/c_4)^2}$$

$$\begin{aligned} &a_1 = 53.4,\ b_1 = 5.165,\ c_1 = 8.815, \\ &a_2 = 31.25,\ b_2 = 18.69,\ c_2 = 5.109, \\ &a_3 = 20.2,\ b_3 = 13.89,\ c_3 = 2.381, \\ &a_4 = 4.316,\ b_4 = 9.864,\ c_4 = 0.992, \end{aligned} \tag{27}$$

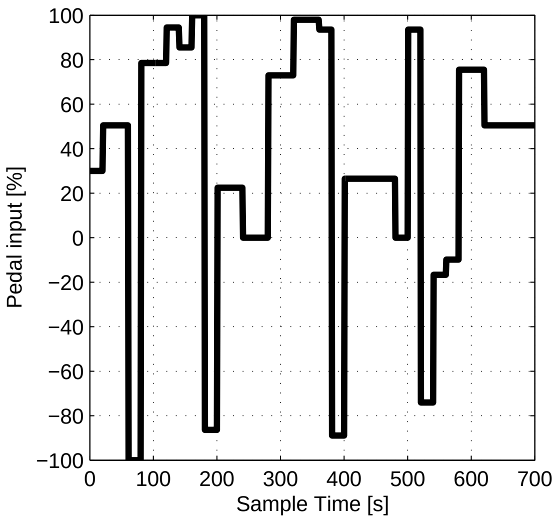
We have made sample data in interval $\{7, 22\}$ (Fig. 5). In our example we did approximation in one point, by two lines. Before parameter estimation it was necessary to normalize data into the interval $\{-1, 1\}$ where Chebyshev polynomials are orthogonal. We used the first four Chebyshev polynomials $T_0 \div T_3$. Mean square error for this approximation is mse $= 5.1947$. To choose a best position of linearization points is a state of art of many algorithms. Through fast network parameters computation it is possible to use even genetic approach to get better position of linearization point and number of chebyshev polynomials.

3.2 *Identification of 3-D function*

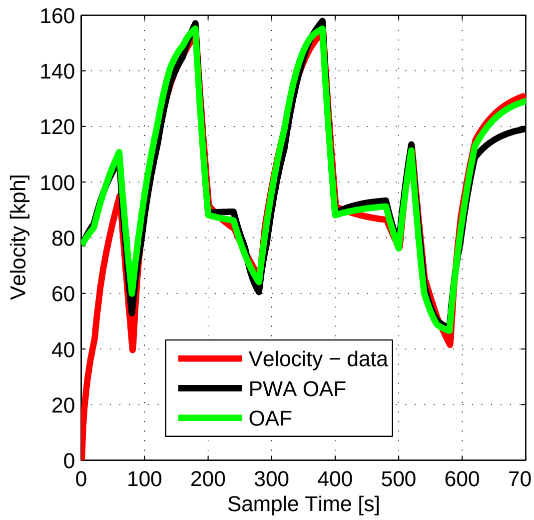
Consider a 3-D nonlinear function defined as

$$\begin{aligned} f(\bar{x}) = &-.2(\sin(x_1 + 4x_2)) - 2\cos(2x_1 + 3x_2) \\ &-3\sin(2x_1 - x_2) + 4\cos(x_1 - 2x_2) \\ &x_1 \in \{0, 1\}, \\ &x_2 \in \{0, 1\}, \end{aligned} \tag{28}$$

We used the first six Chebyshev polynomials, up to the fifth order $T_0 \div T_5$, linearized in 1 point, each polynomial by two lines. The total number of shifting cases for the resulting PWA function is n_u^{lp+1} where n_u is the number of neural network inputs and lp is the number of linearization points.



(a) Pedal time response



(b) Vehicle speed time response

Fig. 6. Vehicle identification data

For the 3-D function example (28) we get $2^2 = 4$ shifting cases. The result is plotted in Fig. 5b. For this approximation mse=0.0144.

3.3 *Modeling and control of nonlinear dynamic system*

In next example we will try to capture vehicle nonlinear dynamic from input output data for purpose of predictive control design of automatic cruise control. We used Simulink vehicle model with automatic transmission controller (Veh, 2006). Input for model is throttle and break torque signal. Output is vehicle velocity. From the character of input signals we can merge throttle and break torque signal to one input signal (Fig. 6a). Positive part of the input signal is proportional to accelerator pedal pressing and negative part of the input signal is proportional to breaking pedal pressing. Input-output data and identified system output are captured in Fig. 6. We used following identification parameters:

$$\begin{aligned} na &= 1 \\ nb &= 1 \\ nk &= 1 \\ npoly &= 4 \quad \text{polynomials:} T_0, T_1, T_2, T_3 \\ ndiv &= 1 \quad \text{approximation by two lines} \end{aligned} \tag{29}$$

These parameters leads to state space model with one state variable and one input. Acquired PWA state space model has four dynamics (four shifting cases) and it is possible to design an automatic cruise control for such system.
For control design we used MPT toolbox (Kvasnica et al., 2004). We designed explicit mpc controller with time varying reference tracking property. We choosed quadratic cost control problem:

$$\min_{u(0),\ldots,u(N-1)} = x(N)^T P_N x(N) + \sum_{k=1}^{N-1} u(k)^T R u(k) + x(k)^T Q x(k) \tag{30a}$$

$$s.t.: \begin{cases} x(k+1|t) = f_{dyn}(x(k), u(k)) \\ u_{min} \le u(k) \le u_{max} \\ \Delta u_{min} \le u(k) - u(k-1) \le \Delta u_{max} \\ y_{min} \le g_{dyn}(x(k), u(k)) \le y_{max} \\ x(N) \in T_{set} \end{cases} \tag{30b}$$

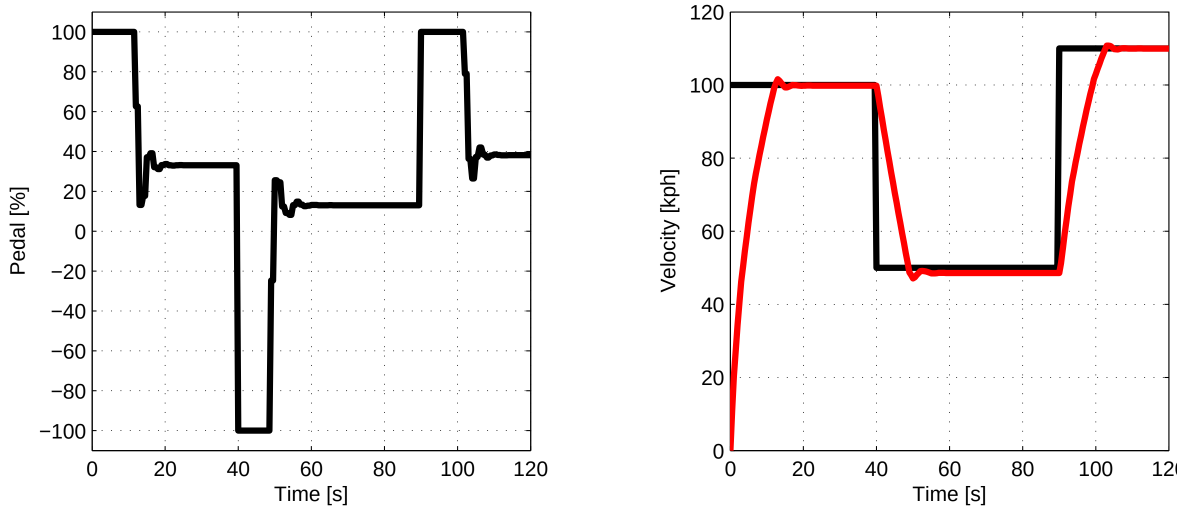
Parameters of control design:

```
       norm: 2
 subopt_lev: 0
          N: 3
   tracking: 1
          Q: 100
          R: 1
         Qy: 700
```

Thanks to few PWA dynamics it is possible choose higher prediction horizon to refine control performance. Resulting control law is defined over 430 regions. It is possible to get satisfactory performance with control law defined over fewer number of regions. Designed control law was used in feedback control with nonlinear vehicle model Fig. 7b.

4. CONCLUSION

PWA OAF toolbox significantly improves identification and modeling of nonlinear systems. Transformation to PWA state space model allows to use existing control design tools. So far PWA OAF ID supports only MISO systems. Three studied cases were presented. It was shown that the proposed



(a) Control and reference input time response

(b) Controlled output time response

Fig. 7. Automatic cruise control

approach was effective in model precision and universal in various input configuration. Computation of network parameters is fast and it allows to execute identification for various parameters (order of used Chebyshev polynomials, number of linearization points) to get better performance or even to use genetic approach. Accuracy of the PWA OAF NN approximation depends on the number of linearization points, the highest order of used Chebyshev polynomials and absolute value of computed parameters of the neural network. More linearization points give better precision of the approximation but complexity of the PWA model increases. It is necessary to find suitable proportion between the number of linearization points and required precision.

ACKNOWLEDGMENTS

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