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LONGITUDINAL H_∞ REDUCED ORDER FLIGHT CONTROL IN A WINDSHEAR

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Abstract: This paper presents the solution for longitudinal flight control problem in a windshear by means of H_∞ -suboptimal controller of given order. The control aims at minimizing $\|T_{zw}\|_\infty$ -norm between wind disturbance and aircraft airspeed and altitude. Comparison of H_∞ -suboptimal reduced order controllers of closed-loop system simulation results is carried out.

Keywords: H_∞ -suboptimal control, reduced order design, flight control, linear matrix inequalities (LMI).

1. INTRODUCTION

Low-altitude windshear with big gradients of vertical and horizontal wind components conditioned by local atmosphere distortion is very dangerous meteorological effect for flying aircrafts. The windshear can commonly appear in area of microburst being compact but sufficiently intensive downdraught. Usually, the microburst spreads to altitude of several hundred meters and has diameter of 5-8 kilometers. Unexpectedly appearing atmosphere distortions are very dangerous for landing aircrafts.

Longitudinal flight control problem aimed to wind disturbance attenuation for aircraft in landing approach along glidepath with relative slope angle was solved by means of H_∞ -suboptimal feedback-loop linear controller of given order. The obtained control law minimizes the influence of windshear on deviations of airspeed and altitude from prescribed values. Angle of attack and thrust are considered as aircraft control.

The paper is organized as follows. Section 2 is devoted to the mathematical model of aircraft longitudinal motion in presence of wind disturbance. Section 3 contains the H_∞ -optimization problem statement for system. Section 4 gives some necessary background for H_∞ theory, as well as makes a

brief mention of analytical solution and numerical algorithm for the control problem. In Section 5, we present the simulation results for landing approach of aircraft controlled by H_∞ -suboptimal controller of a different orders in a windshear together. A number of concluding remarks is given in Section 6.

2. MATHEMATICAL MODEL OF AIRCRAFT MOTION

Longitudinal motion of an aircraft subject to wind disturbance in speed reference frame (tangent and normal lines to flying path) with the assumption of constant mass can be described by the following equations

$$\left. \begin{aligned} m\dot{V} &= T \cos \alpha - D - mg \sin \theta \\ &\quad - m(\dot{w}_x \cos \theta + \dot{w}_y \sin \theta), \\ mV\dot{\theta} &= T \sin \alpha + L - mg \cos \theta \\ &\quad + m(\dot{w}_x \sin \theta - \dot{w}_y \cos \theta), \\ J_z \dot{\omega}_z &= M_z, \\ \dot{\vartheta} &= \omega_z, \end{aligned} \right\} \quad (1)$$

where m is aircraft mass, V is airspeed, T is thrust, α is angle of attack, D is drag force, g is acceleration of gravity, θ is flying path slope angle, \dot{w}_x and \dot{w}_y are full gradients of horizontal

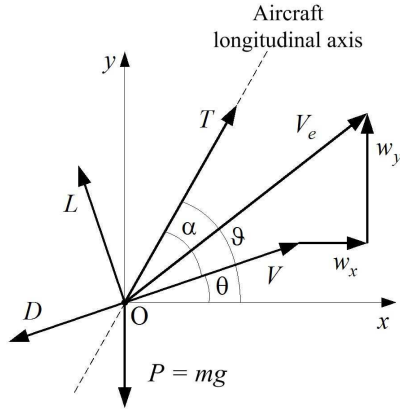


Fig. 1. Aircraft longitudinal flight: reference frame and variables

and vertical wind speed components in inertial reference frame, respectively, L is lift force, J_z is moment of inertia with respect to aircraft z -axis, ω_z is angular speed with respect to aircraft z -axis, M_z is pitching moment, and $\vartheta = \alpha + \theta$ is angle of pitch (see figure (1)).

The thrust T and the angle of attack α are the control variables in equations (1) that depend on deflections of throttle lever δ_t and aircraft generalized elevator δ_e , respectively. So, aircraft control in longitudinal plane is realized via generalized elevator δ_e and throttle lever δ_t .

Differential equation for aircraft mass center altitude is given by

$$\dot{h} = V \sin \theta + w_y. \quad (2)$$

Engine dynamics is described by the following equation

$$\Delta \dot{T} = \frac{1}{T_e} (-\Delta T + K_e \Delta \delta_t), \quad (3)$$

where T_e is engine response time, K_e is some numerical coefficient, and $\Delta \delta_t$ is throttle lever deflection from prescribed value.

Generalized elevator deflection $\Delta \delta_e$ subject to short-period motion loop is formed in the following way

$$\Delta \delta_e = K_{\omega_z} \Delta \omega_z + K_{\vartheta} \Delta \vartheta + K_{cy} \Delta \vartheta_{cy},$$

where K_{ω_z} , K_{ϑ} , and K_{cy} are some numerical coefficient, and $\Delta \vartheta_{cy}$ is control signal generated by controller.

3. PROBLEM STATEMENT

In neighborhood of given glidepath, nonlinear model (1)-(3) of aircraft motion can be approximated by linearized mathematical model. Consider linear discrete time-invariant system F:

$$\left. \begin{aligned} x_{t+1} &= Ax_t + B_1 w_t + B_2 u_t, \\ z_t &= C_1 x_t + D_{11} w_t + D_{12} u_t, \\ y_t &= C_2 x_t + D_{21} w_t + D_{22} u_t. \end{aligned} \right\} \quad (4)$$

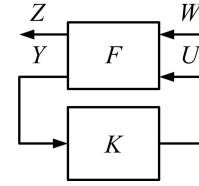


Fig. 2. Block-diagram of closed-loop system

where $x_t \in \mathbb{R}^n$ is the state, $z_t \in \mathbb{R}^{n_z}$ is the controlled output, $u_t \in \mathbb{R}^{n_u}$ is the control, $w_t \in \mathbb{R}^{n_w}$ is the disturbance, $y_t \in \mathbb{R}^{n_y}$ is the observation:

$$\begin{aligned} x_t &= [\Delta V_t, \Delta \theta_t, \Delta \omega_{z,t}, \Delta \vartheta_t, \Delta h_t, \Delta T_t]^T, \\ w_t &= [w_{y,t}, \dot{w}_{x,t}, \dot{w}_{y,t}]^T, \\ u_t &= [\Delta \theta_{cy,t}, \Delta \delta_{t,t}]^T, \\ z_t &= [\Delta V_t, \Delta h_t, \Delta \vartheta_{cy,t}, \Delta \delta_{t,t}]^T, \\ y_t &= [\Delta V_t, \Delta h_t]^T; \end{aligned}$$

$A, B_0, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22}$ are known matrices of appropriate dimensions.

Consider the system F closed by admissible controller K being strictly causal linear discrete time-invariant system that internally stabilizes closed-loop system with transfer matrix $T_{zw}(z) = \mathcal{L}(F, K)$ from disturbance input $W = (w_t)$ to controlled output $Z = (z_t)$ (that is $T_{zw} \in H_\infty^{n_z \times n_w}$) (see figure (2)).

Let us formulate the H_∞ -suboptimal reduced order controller design problem for plant (4):

Problem. For given system (4) find a controller K of order $k \geq n$ satisfying the following conditions:

$$\|T_{zw}\| < \gamma, \forall w, \|w\| \neq 0 \quad (5)$$

4. BACKGROUND

Consider controllable linear discrete object given as

$$\begin{aligned} x_{t+1} &= Ax_t + B_1 w_t + B_2 u_t, \\ z_t &= C_1 x_t + D_{11} w_t + D_{12} u_t, \\ y_t &= C_2 x_t + D_{21} w_t + D_{22} u_t. \end{aligned} \quad (6)$$

where $x_t \in \mathbb{R}^{n_x}$ is the state space; $w_t \in \mathbb{R}^{n_w}$ - disturbance; $u_t \in \mathbb{R}^{n_u}$ - control input; $z_t \in \mathbb{R}^{n_z}$ and $y_t \in \mathbb{R}^{n_y}$ - controllable and measure output respectively. The problem is to construct linear dynamic controller of order k

$$\begin{aligned} x_{t+1}^{(r)} &= A^{(r)} x_t^{(r)} + B^{(r)} y_t, \\ u_t &= C^{(r)} x_t^{(r)} + D^{(r)} y_t, \end{aligned} \quad (7)$$

where $x_t^{(r)} \in \mathbb{R}^k$ - state of the controller, which provide asymptotically stability of closed loop system (6),(7) satisfying the following conditions

$$\frac{\|z\|}{\|w\|} < \gamma, \forall w, \|w\| \neq 0 \quad (8)$$

for given γ .

Denote the controller matrix as

$$\Theta = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \in \mathbb{R}^{(n+k) \times (n+k)} \quad (9)$$

Subject to (9) the closed loop system (6),(7) can be described as

$$\begin{aligned} A_{cl} &= A_0 + \mathcal{B}\Theta\mathcal{C}, & B_{cl} &= B_0 + \mathcal{B}\Theta\mathcal{D}_{21} \\ C_{cl} &= C_0 + \mathcal{D}_{12}\Theta\mathcal{C}, & D_{cl} &= D_{11} + \mathcal{D}_{12}\Theta\mathcal{D}_{21} \end{aligned} \quad (10)$$

where

$$A_0 = \begin{pmatrix} A & 0_{n_x \times k} \\ 0_{k \times n_x} & 0_{k \times k} \end{pmatrix}, B_0 = \begin{pmatrix} B_1 \\ 0_{k \times n_w} \end{pmatrix}, \quad (11)$$

Proposition 1 [7] Consider a discrete-time transfer function $T(z)$ of minimal realization $T(z) = D + C(zT - A)^{-1}B$. The H_∞ -controller of order k exist if and only if there exist two mutually inverse matrices X and Y ($XY = I, X, Y \in \mathbb{R}^{(n_x+k) \times (n_x+k)}$) satisfying the following LMIs

$$\begin{aligned} W_P^T \begin{pmatrix} -Y & A_0 & B_0 & 0 \\ A_0^T & -X & 0 & C_0^T \\ B_0^T & 0 & -\gamma I & D_{11}^T \\ 0 & C_0 & D_{11} & -\gamma I \end{pmatrix} W_P &< 0 \\ W_Q^T \begin{pmatrix} -Y & A_0 & B_0 & 0 \\ A_0^T & -X & 0 & C_0^T \\ B_0^T & 0 & -\gamma I & D_{11}^T \\ 0 & C_0 & D_{11} & -\gamma I \end{pmatrix} W_Q &< 0 \end{aligned} \quad (12)$$

If conditions (12) are met and matrices X and Y are found, then parameters of the controller Θ can be found as a solution of LMI $\Psi + P^T\Theta^TQ + Q^T\Theta P < 0$. Here

$$\begin{aligned} \Psi &= \begin{pmatrix} -Y & A_0 & B_0 & 0 \\ A_0^T & -X & 0 & C_0^T \\ B_0^T & 0 & -\gamma I & D_{11}^T \\ 0 & C_0 & D_{11} & -\gamma I \end{pmatrix} \\ \mathcal{Q} &= \begin{pmatrix} \mathcal{B} & 0_{(n_u+k) \times (n_u+k)} & 0_{(n_u+k) \times (n_w)} & \mathcal{D}_{12}^T \\ \mathcal{P} & 0_{(n_y+k) \times (n_x+k)} & \mathcal{C} & \mathcal{D}_{21}^T \end{pmatrix} \begin{pmatrix} 0_{(n_u+k) \times (n_u+k)} \\ 0_{(n_y+k) \times (n_z)} \end{pmatrix} \end{aligned} \quad (13)$$

Mutual inverse matrices search algorithm

The problem of finding mutually inverse matrices can be formulated as

$$\begin{aligned} \text{Find } \lambda_{min} &= \min(\lambda : XY^{-1} < \lambda I, \\ X > 0, Y > 0, L_i(X, Y) < 0, i &= 1, 2, 3) \end{aligned} \quad (14)$$

where

$$L_3(X, Y) = \begin{pmatrix} -X & I \\ I & -Y \end{pmatrix}.$$

$L_1(X, Y)$ and $L_2(X, Y)$ are defined in (12). An additional LMI $L_3(X, Y) < 0$ equivalent to LMI $X - Y^{-1} < \lambda I$ (by Schur's lemma). Thus $\lambda_{min} = 0$ implies $X = Y^{-1}$.

Minimization of objective linear function with constraints is required for solve the problem. One of them is

$$XY^{-1} < \lambda I \quad (15)$$

This constraint is non convex, Therefore, it can not be represent as a linear matrix inequality. This fact make it impossible to solve the problem by means of convex optimization. The following algorithm is able to solve the problem using MATLAB LMI Control Toolbox.

For the description of this algorithm define the following optimization problem [1]:

$$\begin{aligned} \text{Find } \lambda_{min} &= \min(\lambda : \Gamma(X, Y < G_1, G_2) < \lambda I, \\ X > 0, Y > 0, L_i(X, Y) < 0, i &= 1, 2, 3) \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Gamma(X, Y < G_1, G_2) &= \\ &= (I \ G_1) \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \begin{pmatrix} I \\ G_1 \end{pmatrix} + \\ &+ (G_2 \ I) \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \begin{pmatrix} G_2 \\ I \end{pmatrix} \end{aligned}$$

$G_i = G_i^T, i = 1, 2$ - are given matrices.

Note that in the problem (16) constraint (15) is replaced by LMI $\Gamma(X, Y < G_1, G_2) < \lambda I$. As

$$\begin{aligned} \Gamma(X, Y < G_1, G_2) &= \\ &= (G_1 + Y^{-1})Y(G_1 + Y^{-1}) + \\ &+ (G_2 + X^{-1})X(G_2 + X^{-1}) + \\ &+ (X - Y^{-1}) + (Y - X^{-1}) \geq 0 \end{aligned} \quad (17)$$

and under the inequality $L_3(X, Y) < 0$ $X > Y^{-1}$ satisfied when $\lambda_{min} = 0$. This implies that the solution (X, Y) of problem (16) is the solution of (14) (here $G_1 = -Y^{-1}$ and $G_2 = -X^{-1}$)

Mutual inverse matrices search algorithm

- (1) Setting $j = 0$.
- (2) Specifying matrices $G_1 = G_1^{(j)}$ and $G_2 = G_2^{(j)}$.
- (3) Solving (16) problem using MATLAB LMI Control Toolbox, finding λ_{j+1}, X_j, Y_j .
- (4) Setting $G_1^{j+1} = -Y_j^{-1}, G_2^{j+1} = -X_j^{-1}, j = j + 1$. Go to step 2.

Proposition 2 [1] For any initial conditions $G_1^{(0)}$ and $G_2^{(0)}$ sequence λ_j generated by algorithm is nondecreasing. Also there are finite limits

$$\lim_{j \rightarrow \infty} (\lambda_j) = \lambda_* \geq 0, \lim_{j \rightarrow \infty} (X_j) = X_*, \lim_{j \rightarrow \infty} (Y_j) = Y_*$$

Proof

Let us estimate a spectral radius of $\Gamma(X, Y, G_1, G_2)$ along the trajectory of algorithm. Represent

$$\Delta\rho = \rho(\Gamma(X_{j+1}, Y_{j+1}, G_1^{j+1}, G_2^{j+1})) - \rho(\Gamma(X_j, Y_j, G_1^j, G_2^j))$$

as

$$\begin{aligned} \Delta\rho &= \Delta\rho_1 + \Delta\rho_2 = \\ &= [\rho(\Gamma(X_{j+1}, Y_{j+1}, G_1^{j+1}, G_2^{j+1})) - \\ &\rho(\Gamma(X_j, Y_j, G_1^{j+1}, G_2^{j+1}))] + \\ &+ [\rho(\Gamma(X_j, Y_j, G_1^{j+1}, G_2^{j+1})) - \\ &\rho(\Gamma(X_j, Y_j, G_1^j, G_2^j))] \end{aligned}$$

The first bracketed expression is nonpositive by operation of algorithm, whereas λ reaches its minimum at $(j + 1)$ -s iteration when $X = X_{j+1}$, $Y = Y_{j+1}$. It follows from (17) that

$$\begin{aligned} \Gamma(X_j, Y_j, G_1^{j+1}, G_2^{j+1}) - \Gamma(X_j, Y_j, G_1^j, G_2^j) = & \\ = (G_1^{j+1} + Y_j^{-1})Y_j(G_1^{j+1} + Y_j^{-1}) + & \\ + (G_2^{j+1} + X_j^{-1})X_j(G_2^{j+1} + X_j^{-1}) - & \\ - (G_1^j + Y_j^{-1})Y_j(G_1^j + Y_j^{-1}) - & \\ - (G_2^j + X_j^{-1})X_j(G_2^j + X_j^{-1}) & \end{aligned}$$

Taking into consideration that $G_1^{j+1} = -Y_j^{-1}$ and $G_2^{j+1} = -X_j^{-1}$ we get

$$\begin{aligned} \Gamma(X_j, Y_j, G_1^{j+1}, G_2^{j+1}) - \Gamma(X_j, Y_j, G_1^j, G_2^j) = & \\ = -(Y_j^{-1} + Y_{j-1}^{-1})Y_j(Y_j^{-1} + Y_{j-1}^{-1}) - & \\ -(X_j^{-1} + X_{j-1}^{-1})X_j(X_j^{-1} + X_{j-1}^{-1}) & \end{aligned}$$

As $A - B \leq 0$ therefore $\rho(A) \leq \rho(B)$, then $\Delta\rho \leq 0$. In this case sequence ρ_j is bounded below and nonincreasing. Therefore limits declared in proposition 2 exist.

There are two situation.

- (1) if $\lambda_* = 0$ then $X_*Y_* = I$. X_*, Y_* is the solution of (14)
- (2) if $\lambda_* > 0$, then the problem (14) may have no solutions. It is recommended to repeat the algorithm with another initial conditions G_1^0 and G_2^0

Algorithm can be stopped if $\lambda_j < \epsilon$ or $|\lambda_{j+1} - \lambda_j| < \epsilon$. It follows from proposition 2 that algorithm stops after finite iteration count.

H_∞ -suboptimal reduced order controller synthesis algorithm

Based on previous facts consider the algorithm which can be applied to solve H_∞ -synthesis problem.

For given discrete linear system (6) the structure of controller is specified (i.e. its order k). As the major condition of existence of controller of order k is the existence of mutual inverse matrices, then algorithm has the following structure:

- (1) The desirable controller's order k is fixed.
- (2) The mutual inverse matrices search algorithm is implemented. It satisfies the following conditions:

$$\text{Find } \lambda_{min} = \min(\lambda : \Gamma(X, Y < G_1, G_2) < \lambda I, X > 0, Y > 0, L_i(X, Y) < 0, i = 1, 2, 3)$$

where

$$\begin{aligned} \Gamma(X, Y < G_1, G_2) = & \\ = (I \ G_1) \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \begin{pmatrix} I \\ G_1 \end{pmatrix} + & \\ + (G_2 \ I) \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \begin{pmatrix} G_2 \\ I \end{pmatrix} & \end{aligned}$$

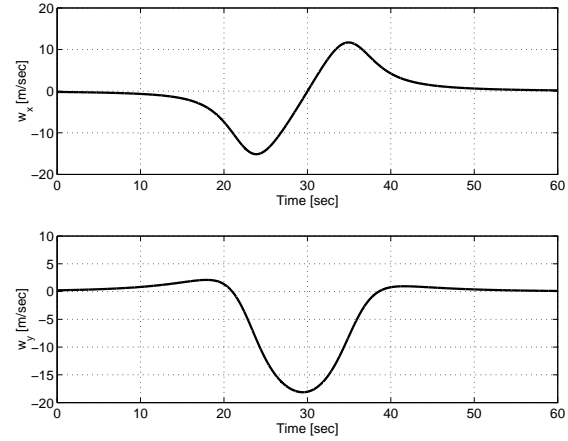


Fig. 3. Deterministic horizontal (w_x) and vertical (w_y) components of wind profile

$$\begin{aligned} L_1(X, Y) = W_P^T \begin{pmatrix} -Y & A_0 & B_0 & 0 \\ A_0^T & -X & 0 & C_0^T \\ B_0^T & 0 & -\gamma I & D_{11}^T \\ 0 & C_0 & D_{11} & -\gamma I \end{pmatrix} W_P < 0 \\ L_2(X, Y) = W_Q^T \begin{pmatrix} -Y & A_0 & B_0 & 0 \\ A_0^T & -X & 0 & C_0^T \\ B_0^T & 0 & -\gamma I & D_{11}^T \\ 0 & C_0 & D_{11} & -\gamma I \end{pmatrix} W_Q < 0 \\ L_3(X, Y) = \begin{pmatrix} -X & I \\ I & -Y \end{pmatrix} \end{aligned}$$

- (3) If X and Y exist, then the controllers parameters can be found as a solution of LMI

$$\Psi + P^T \Theta^T Q + Q^T \Theta P < 0$$

where

$$\Theta = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$$

and P, Q are defined in (13). If solution X, Y at step 2 cannot be found, then go to step 1 and change k .

5. SIMULATION RESULTS

The H_∞ -suboptimal controller design problem was solved for aircraft *TU-154* in landing approach along glidepath with fixed relative slope angle. Aircraft longitudinal motion equations (1)-(3) were linearized in the trajectory point with the parameters $V_0 = 71.375$ m/sec, $h_0 = 600$ m, and $\theta_0 = -2.7$ deg.

Table 1. H_∞ -norms of closed-loop systems with controllers of different order

Order (k)	6	5	4	3
$\ T_{zw}\ _\infty$	17.50	17.25	17.38	17.09
Order (k)	2	1	0	
$\ T_{zw}\ _\infty$	16.92	17.44	17.43	

Table (1) demonstrate us norms of closed loop system with controllers of different orders. As it can be seen, controllers of different order provides

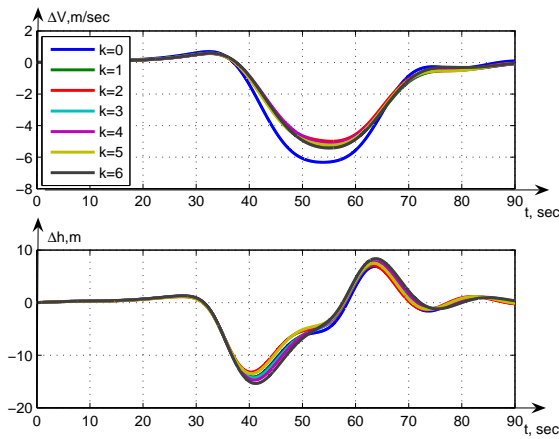


Fig. 4. Controlled variables z_1 (ΔV , m/sec) and z_2 (Δh , m)

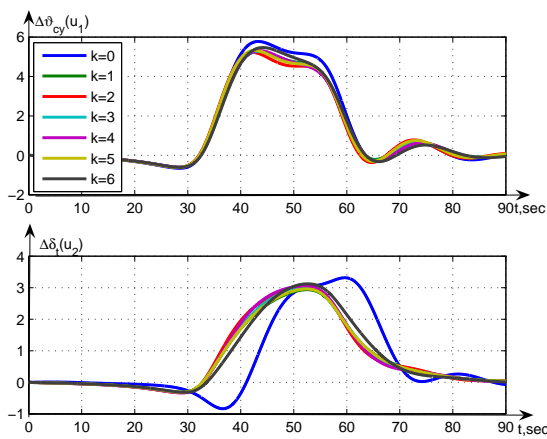


Fig. 5. Control variables u_1 ($\Delta\theta_{cy}$, deg) and u_2 ($\Delta\delta_t$, deg)

almost the same performance. This means that it is very useful to use the controller of simplified realization.

6. CONCLUSION

This paper presents the solution for longitudinal flight control problem in a windshear by means of H_∞ -suboptimal controller of given order. The simulation for aircraft in landing approach along glidepath with fixed relative slope angle shows that the static output feedback controller provides internal stability of closed-loop system and its norm is close to that one closed by means of full-order controller. The comparison between H_∞ -suboptimal reduced order controllers show us that mutual inverse matrices search algorithm is very useful for some practical application such as flight control.

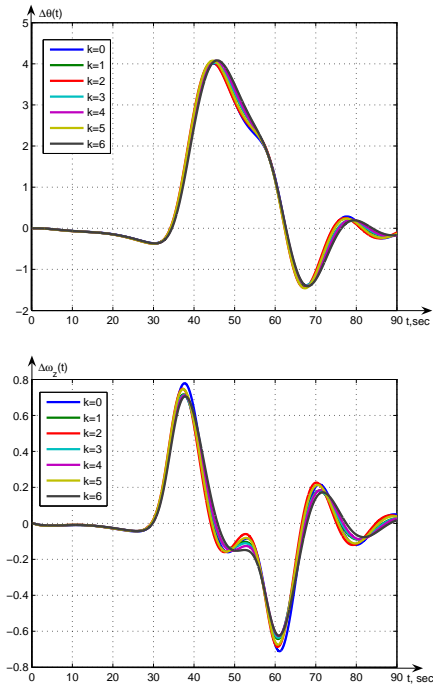


Fig. 6. State variables ($\Delta\theta$, deg) and ($\Delta\omega_z$, deg/sec)

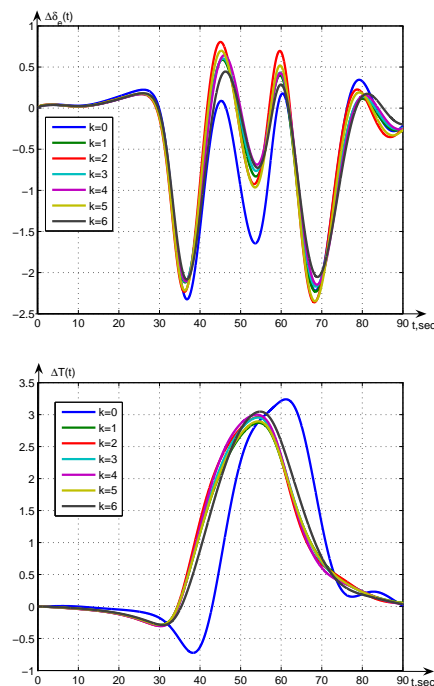


Fig. 7. State variables ($\Delta\delta_e$, deg) and (ΔT , kN)

References

- [1] D.V. Kogan, M.M. Balandin. LMI based control laws synthesis. M.:Phismatgis, 2007.
- [2] A.P. Kurdyukov, I.G. Vladimirov, V.N. Timin et al. Methods of Classical and Modern Control Theory: 3 volume textbook. Vol.3: Methods of Modern Control Theory. M.:BMSTU, Moscow, 2000.

- [3] A.P. Kurdyukov, V.N. Timin. *Longitudinal Robust Flight Control Synthesis in a Windshear*. Technical Cybernetics,6, 1993.
- [4] J.C. Doyle, K. Glover, P.P. Khargonekar, B.A. Francis. *State-Space Solution to Standart H_2 and H_∞ Control Problems*. IEEE Trans. of Automatic Control, AC-34 #8, 1989.
- [5] P. Gahinet. *Explicit Controller Formulas for LMI-based H_∞ Synthesis*. submitted to Automatica. Also in Proc. Amer. Contr. Conf., pp. 2396-2400, 1994.
- [6] P. Gahinet, P. Apkarian. *A Linear Matrix Inequality Approach to H_∞ Control*. Int. J. Robust and Nonlinear Control, pp. 421-448, 4 (1994).