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ON STABILITY TESTS OF SPATIALLY DISTRIBUTED SYSTEMS

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Abstract: The paper describes tests of stability of spatially distributed shift-invariant systems discrete in both time and space. The systems are considered to be described by multivariate polynomial fractions, so, the tests based on manipulation with polynomials are taken into account. Method of root maps is depicted. Methods based on the Schur-Cohn criterion, originally formulated for systems with lumped parameters, are extended to multidimensional systems with support on a symmetric half-plane. Furthermore, the problem of stability of multivariate polynomial is formulated as a problem of stability of interval polynomial, which leads to use Kharitonov's theorem. Numerical examples are included.

Keywords: Spatially distributed shift-invariant systems, multidimensional systems, multivariate polynomials, stability.

1. INTRODUCTION

The control of *spatially distributed systems* has always been a very active topic with applications in many areas, e. g. image processing, multidimensional filtering techniques, long-wall coal cutting and metal rolling, irrigation canals in agriculture, adaptive optics, etc. These systems are mathematically described by partial differential equations (PDEs). Considering using mesh of sensors and actuators, one possible approach to the problem of the control of such systems is spatial discretisation of (linearised) PDE and description of the dynamics by a transfer function or a state-space model of multidimensional (n-D) systems theory.

We consider a spatially distributed system described by transfer function which was derived in form

$$P = \frac{b(z, z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})}{a(z, z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})}, \tag{1}$$

i. e. in the form of fraction of two multivariate (n-D) polynomials a and b. The variable z corresponds to time delay, while the variables $z_1, \ldots, z_n$ correspond to shift along the spatial coordinate axis. The system (1) belongs to so-called systems *with support on a symmetric half-plane.* Polynomials a and b are one-sided in z and two--sided in $z_1, \ldots, z_n$.

We suppose that the system is spatially symmetric, i. e. the polynomial a has the form

$$a = \sum_{i,i_1,\ldots,i_n} a_{i,i_1,\ldots,i_n} z^i \left(z_1^{i_1} + z_1^{-i_1}\right) \cdots \left(z_n^{i_n} + z_n^{-i_n}\right) \tag{2}$$

with $i, i_1, \ldots, i_n$ nonnegatives.

Furthermore, the plant is considered to be spatially invariant and infinite. This assumption must come true to perform z-transform to obtain transfer function and allows not to take into account the boundary conditions of the system.

It is well known that necessary and sufficient conditions for asymptotic stability of n-D systems can be described in terms of an n-D characteristic polynomial. Unlike 1-D systems, stability of n-D systems depends generally both on denominator and numerator of transfer function. However, we will not be concerned in such the cases and we refer an interested reader to Jury (1978) or Dudgeon and Mersereau (1984) for whole information. The following lemma holds.

Lemma 1. A system with transfer function (1) is stable if and only if

$$a \neq 0, \quad \left\{ \bigcap_{i=1}^{n} |z_i| = 1 \right\} \cap \{|z| \geq 1\} . \tag{3}$$

The lemma means that polynomial a must not have any root on unit polycircle for all variables corresponding to shift along the spatial coordinate axis and within the unit polydisc for variable corresponding to time delay. See Jury (1978) for details and references to papers containing the proof of the above lemma.

Stability of n-D systems is still interesting topic, see e. g. papers by Huang (1972), Šiljak (1975), Serban and Najim (2007) or proceedings by Henrion and Garulli (2005). However, most of papers deals with criteria for systems *with support on a quarter-plane*, i. e. with stability region,

$$a \neq 0, \quad \left\{ \bigcap_{i=1}^{n} |z_i| \geq 1 \right\} \cap \{|z| \geq 1\} ,$$

Stability region (3) is considered by Jury (1978) and Bose (1985) but a few methods are purposed to test the stability.

Remark 2. Note that substituting $z_i = e^{j\omega_i}$ we can write the criterion (3) in the form

$$a(z, e^{j\omega_1}, \dots, e^{j\omega_n}) \neq 0, \quad \{\forall \omega_i \in \mathbb{R}\} \cap \{|z| \geq 1\} .$$

The Lemma 1 leads straightforwardly to the following corollary.

Corollary 3. The transfer function (1) can be considered to be fraction of elements of a ring

$$\mathcal{R}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}][z],$$

i. e. a fraction of two polynomials in one indeterminate z with coefficients defined in terms of $z_1, z_1^{-1}, \dots, z_n, z_n^{-1}$. Such a description can be used to study stability and stabilizability.

This paper deals with a few methods which are able to check the criterion (3). Method based on root maps is described first. It is straightforward extension to computing of roots of univariate (1--D) polynomial. Then methods based on Schur--Cohn type criterion known from 1-D system theory are extended to check the region (3). Finally, stability of n-D polynomial is reformulated as stability of interval 1-D polynomial and Kharitonov's theorem is then used to decide on system stability.

The paper is organised as follows. The methods mentioned above are described in the next section. One of them leads to use of checking polynomial positivity. Thus, in Sec. 3 we introduce possible techniques how to check positivity of a polynomial matrix. Sec. 4 contains a number of examples on stability tests. At the end of the paper concluding remarks are made.

2. STABILITY CRITERIA

A short description of couple of criteria existing for stability of 1-D systems is given. Having in mind the Corollary 3, these criteria are extended to analyse stability of n-D systems.

2.1 *Location of poles in the complex z-plane*

One of the most trivial way how to decide on stability is direct use of Lemma 1 by computing roots of the polynomial a. A system is stable if and only if all roots lie within the stability region. While in 1-D case the roots are isolated points, given by solution to $a(z) = 0$, in n-D case a situation is more complicated. The roots are curves. Their location can be determined using so-called *root map* — 2-D graph consisting of n parts, where each part shows the loci of the roots of $a[z_1, \dots, z_{i-1}, \ z_{i+1}, \dots z_n](z_i)$ as the parameters $z_1, \dots, z_{i-1}, \ z_{i+1}, \dots z_n$ traverse the unit circle $z_k = e^{j\,\omega_k}$ for $-\pi \leq \omega_k \leq \pi$, $k \neq i$, for $i = 1, 2, \dots, n$, see e. g. Dudgeon and Mersereau (1984). The following lemma holds, see Augusta et al. (2007) for details.

Lemma 4. A system given by (1) is stable if and only if its root map generated by $a[z_1, \dots, z_n](z)$ lies inside the unit circle in the z-plane.

This lemma says only root map generated by $a[z_1, \dots, z_n](z)$, where z corresponds to time delay, must be plotted to decide on stability.

2.2 *Schur-Cohn type criterion*

The fact that we do not have to know the exact location of poles to decide on system stability and all we need to know is if there is a pole which lies outside the stability region motivates us to use the Schur-Cohn type criterion, see Barnett (1983), in 1-D case as well as in n-D case.

In 1-D case, consider the transform Φ that maps a complex function F to the function $\Phi(F)$ defined by

$$\Phi(F)(z) = \begin{cases} 0 & F \text{ is a constant function of modulus equal to 1} \\ \dfrac{F(z) - F(0)}{z(1 - \overline{F(0)}F(z))} & z \neq 0 \\ F'(0)(1 - |F(0)|^2)^{-1} & z = 0, \end{cases}$$

see e. g. Serban and Najim (2007). Associate the sequence of function $(F_k)_{k=0,1,\ldots}$ to a function F using recursion

$$F_0 = F, \qquad F_k = \Phi\left(F_{k-1}\right), \ k \geq 1.$$

The functions F_k are called *the Schur iterates* of F and parameters

$$\gamma_k = F_k(0), \quad k \geq 1$$

the Schur coefficients of F.

Let $\overline{a(z)}$ be a polynomial of degree m, $\tilde{a}(z) = z^m\, \overline{a(1/z)}$ and $F = \frac{a}{\tilde{a}}$. A polynomial a has no zeros outside the closed unit circle if and only if

$$|\gamma_k| < 1, \qquad k = 0, \ldots, m-1.$$

The basic principle will be demonstrated by means of an example.

Example 5. Consider a system with the transfer function

$$S(z) = \frac{1}{\left(z + \frac{1}{2}\right)\left(z - \frac{1}{8}\right)}. \tag{4}$$

The function $F = \frac{a}{\tilde{a}}$ is equal to

$$F = \frac{\left(z + \frac{1}{2}\right)\left(z - \frac{1}{8}\right)}{z^2\left(\frac{1}{z} + \frac{1}{2}\right)\left(\frac{1}{z} - \frac{1}{8}\right)}.$$

Absolute values of the Schur coefficients are

$$|\gamma_0| = \frac{1}{16}, \qquad |\gamma_1| = \frac{2}{5},$$

all less than 1, the system (4) is stable.

In n-D systems case, Schur coefficients are functions of corresponding to space shift variables. Using Corollary 3 we can consider F to be a function of z with coefficients in $z_1, \ldots, z_n$ and state the following.

Denote $\mathbf{z} = (z_1, \ldots, z_n)$. Consider the transform Φ that maps a complex function F to the function $\Phi(F)$ defined by

$$\Phi(F)(\mathbf{z})(z) = \begin{cases} 0 & F \text{ is a constant function of modulus equal to 1} \\ \dfrac{F(\mathbf{z})(z) - F(\mathbf{z})(0)}{z(1 - \overline{F(\mathbf{z})(0)}F(\mathbf{z})(z))} & z \neq 0 \\ F'(\mathbf{z})(0)(1 - |F(\mathbf{z})(0)|^2)^{-1} & z = 0. \end{cases}$$

Associate the sequence of function $(F_k)_{k=0,1,\ldots}$ to a function F using recursion

$$F_0 = F, \qquad F_k = \Phi\left(F_{k-1}\right), \ k \geq 1.$$

The functions F_k are called *the Schur iterates* of F and parameters

$$\gamma_k = F_k(\mathbf{z})(0), \quad k \geq 1$$

the Schur coefficients of F.

Let $a[\mathbf{z}](z)$ be a polynomial in $\overline{z \text{ of degree}}$ m with coefficients in $\mathbf{z}$, $\tilde{a}[\mathbf{z}](z) = z^m\, \overline{a[\mathbf{z}](1/z)}$ and $F = \frac{a}{\tilde{a}}$. A polynomial a has no zeros outside the closed unit polydisc if and only if

$$|\gamma_k(\mathbf{z})| < 1, \qquad k = 0, \ldots, m-1$$

for all values $|z_i| = 1, i = 1 \ldots, n$.

2.3 *Use of Schur-Cohn matrix*

Another approach to Schur-Cohn stability criterion is use of the Schur-Cohn matrix associated to a complex polynomial

$$a(z) = \sum_{i=0}^{m} a_i\, z^i,$$

$D_a = d_{ij}, 1 \leq i, j \leq m$, which has a form

$$d_{ij} = \sum_{k=1}^{i} \left(a_{n-i+k}\, \bar{a}_{n-j+k} - \bar{a}_{i-k}\, a_{j-k}\right), \quad i \leq j.$$

A polynomial $a(z)$ has all its zeros inside the unit circle if and only if the matrix D_a is positive definite.

Example 6. The Schur-Cohn matrix associated to denominator polynomial of (4) reads

$$D_a = \begin{pmatrix} \frac{255}{256} & \frac{51}{128} \\ \frac{51}{128} & \frac{255}{256} \end{pmatrix}.$$

One can make sure that D_a is positive definite and system (4) is stable which is consistent with the result of Example 5.

Multidimensional analog of the above described 1-D case can be made like in the previous section. Consider a complex polynomial

$$a[\mathbf{z}](z) = \sum_{i=0}^{m} \sum_{i_1=-m_1}^{m_1} \cdots \sum_{i_n=-m_n}^{m_n} a_{i,i_1\cdots,i_n}\, z^i\, z^{i_1} \cdots z^{i_n},$$

$D_a = d_{ij}, 1 \leq i, j \leq m$, which has a form

$$d_{ij} = \sum_{k=1}^{i} \left(a_{n-i+k}\, \bar{a}_{n-j+k} - \bar{a}_{i-k}\, a_{j-k}\right), \quad i \leq j.$$

A polynomial $a[\mathbf{z}](z)$ has all its zeros inside the unit polydisc if and only if

$$D_a(\mathbf{z}) \succ 0 \quad \forall z_i, |z_i| = 1, i = 1, \ldots, n. \qquad (5)$$

Stability of a system thus depends on positivity of n-D polynomial matrix. In Sec. 3 we introduce methods how to check whether a polynomial matrix is positive definite or not.

2.4 *Use of Kharitonov's theorem*

In this section, we will consider spatially symmetric systems described by the transfer function (1) with the polynomial a having a form (2). Substitution $z_k = e^{j\,\omega_k}$ for $-\pi \leq \omega_k \leq \pi$, $k = \{i_1, i_2, \ldots, i_n\}$ into $a[z_1, \ldots, z_n](z)$ gives one-sided polynomial in one variable z with real coefficients for all values of ω_i. Using Euler's formula and taking into account that cosine is bounded function we can express a as so-called interval polynomial. Finally, discrete version of the famous Kharitonov's theorem, see Kharitonov (1978), Kraus et al. (1987), which provides a test of stability for discrete-time interval polynomials can be performed and stability of finite number of so-called corner polynomials must be found out.

The basic ideas expressed in the above paragraphs can be represented by diagram

$$(z_1 + z_1^{-1}) \rightarrow (e^{j\omega_1} + e^{-j\omega_1}) \rightarrow 2\cos\omega_1 \rightarrow [-2; 2].$$

In case that the coefficients of interval polynomial are linked (more then one coefficient depend on, for example, ω_1), the stability condition is sufficient, not necessary.

3. POSITIVITY OF N-D POLYNOMIAL MATRIX

The condition (5) can be checked using theory of positive polynomials. In this section we focus on how to check positivity of n-D polynomial matrix. At first we are concerned with the case where $n = 1$, meaning that the transfer function (1) is fraction of polynomials in two variables, the first corresponding to time delay and the second corresponding to space shift. Then we continue with more difficult case where $n \geq 1$.

3.1 *Case $n = 1$*

The first method is based on well-known fact, see e. g. Yakubovich (1970), Ježek and Kučera (1985), Dumitrescu (2007), that the positivity of two-sided polynomial matrix is connected with existence of polynomial spectral factorisation. Dumitrescu (2007) states that a polynomial

$$R(z) = \sum_{k=-n}^{n} r_k\, z^{-k}, \quad r_{-k} = r_k^\star \qquad (6)$$

is nonnegative on the unit circle if and only if a causal polynomial

$$H(z) = \sum_{k=0}^{n} h_k\, z^{-k}$$

exists such that

$$R(z) = H(z)\, H^\star(z^{-1}),$$

where $\star$ denotes complex conjugation. In other words, existence of spectral factorisation of $R(z)$ is equivalent with nonnegativity on the unit circle of $R(z)$. Even thought we need to check the positivity of $R(z)$, the above idea can be useful. Choose a constant $\varepsilon > 0$ and determine if spectral factorisation of $R(z) - \varepsilon$ exists.

Inspired by Bauer's method of spectral factorisation, see e. g. survey paper by Goodman et al. (1997), or Hromčík and Šebek (2006), the test of nonnegativity of $R(z) - \varepsilon$ can be done using corresponding to the polynomial $R(z) - \varepsilon$ Toeplitz's matrix of order N constructed according to the scheme

$$T_N = \begin{pmatrix} \bar{r}_0 & r_1 & \cdots & r_n & 0 & \cdots & 0 \\ r_{-1} & \bar{r}_0 & r_1 & \cdots & r_n & \ddots & \vdots \\ \vdots & r_{-1} & \ddots & \ddots & & \ddots & 0 \\ r_{-n} & \vdots & \ddots & & & & r_n \\ 0 & r_{-n} & & & & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & r_1 \\ 0 & \ldots & 0 & r_{-n} & \ldots & r_{-1} & \bar{r}_0 \end{pmatrix},$$

where $\bar{r}_0 = r_0 - \varepsilon$ and $N \geq n$. If $T_N \succ 0$ then spectral factorisation of $R(z) - \varepsilon$ exists and $R(z)$ is positive polynomial.

Accomplishment of this method depends significantly on size of ε and N. For example, the polynomial $R(z) = z + 2 + z^{-1}$ is obviously not positive on the unit circle, as $R(-1) = 0$. Choose $\varepsilon = 0.1$ and get $R(z) - \varepsilon = z + 1.9 + z^{-1}$, which is not nonnegative, as $R(-1) - \varepsilon = -0.1$. However, corresponding $T_5 \succ 0$, while $T_{15} \not\succ 0$. Since numerical accuracy of spectral factor computed by Bauer's method goes up with growing N, it is clear that N should be chosen enough great.

3.2 *General case*

In general, there is no spectral factor of nonnegative n-D polynomial and hence the idea of the above subsection is useless. A way is to check whether the n-D polynomial matrix $R(\mathbf{z})$ is positive on the unit n-circle directly. In fact, this problem is solved by Dumitrescu (2007). He replaces the test of positivity of $R(\mathbf{z})$ with test whether $R(\mathbf{z}) - \varepsilon$ with $\varepsilon > 0$ is sum-of-squares. If so, $R(\mathbf{z})$ is positive.

We will use an another technique. At first note that the condition on positivity of polynomial matrix $R(\mathbf{z})$ is equivalent with

$$R(\mathbf{1}) \succ 0$$
$$r(\mathbf{z}) = \det R(\mathbf{z}) > 0, \quad \forall z_i, |z_i| = 1, i = 1, \dots, n,$$

see e. g. Serban and Najim (2007) for details. The problem (5) falls into two steps, checking positivity of a constant matrix and checking positivity of a scalar n-D two-sided polynomial on the unit circle. The first step is trivial. The second one can be check as follows.

Since the polynomial $r(\mathbf{z})$ has the form (2), after substitution introduced in Remark 2 and using Euler's formula it becomes in expression consisting of functions cosine of $\omega_i, i = 1, \dots, n$. Cosine is bounded function and riches values within the interval $[-1; 1]$. So, if we are interested in values of polynomial of $(z_i + z_i^{-1}), |z_i| = 1, i = 1, \dots, n$, in fact, we are interested in values $\zeta_i, \zeta_i \in [-2; 2], \zeta_i = z_i + z_i^{-1}, i = 1, \dots, n$.

The above ideas can be expressed as follows. Substitute $z_i + z_i^{-1} = \zeta_i$ into $r(\mathbf{z})$ and find

$$\mu = \min r(\zeta_1, \dots, \zeta_n)$$
$$\text{s. t. } -2 \le \zeta_i \le 2, \quad i = 1, \dots, n.$$

If $\mu > 0$ then $r(\mathbf{z})$ is positive on the unit circle.

4. EXAMPLES

The concept described before will be demonstrated by means of examples in this section. Stability of two systems will be analysed.

4.1 *A heat conduction in a rod*

A heat conduction in a rod with array of temperature sensors and actuators can be described by transfer function

$$P = \frac{b(z, z_1, z_1^{-1})}{a(z, z_1, z_1^{-1})} = \frac{1}{z - \frac{T}{h^2} z_1^{-1} - 1 + 2\frac{T}{h^2} - \frac{T}{h^2} z_1}, \quad (7)$$

where the output is temperature and the input is input heat, T and h denote respectively a time sample period and distance between nodes, z and z_1 correspond respectively to time delay and to shift along the spatial coordinate axis. See Augusta et al. (2007) for details.

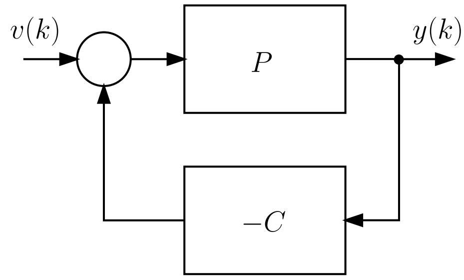

Fig. 1. Control scheme, $v(k)$ — reference signal, $y(k)$ — output

Consider the control scheme of Fig. 1 and a controller with the transfer function

$$C(z, z_1, z_1^{-1}) = c_0 + c_1(z_1 + z_1^{-1}), \quad (8)$$

where c_0 and c_1 are real constants. The characteristic polynomial of closed-loop system has the form

$$\chi(z, z_1, z_1^{-1}) = z + \left(c_1 - \frac{T}{h^2}\right)(z_1 + z_1^{-1}) + c_0 + 2\frac{T}{h^2} - 1. \quad (9)$$

Choose $T = 0.1\,\text{ms}$ and $h = \frac{1}{59}\,\text{m}$ and consider the controller (8) with

$$c_0 = 0.25, \qquad c_1 = 0.25$$

and use the above described methods to find out stability of (9), which now reads

$$z - 0.0981\,(z_1 + z_1^{-1}) - 0.0538. \quad (10)$$

The method of root maps gives the result depicted in Fig. 2. Since the root map lies inside the unit circle, the closed-loop system is stable.

If we use the method of Schur-Cohn coefficients described in Sec. 2.2 we have $m = 1$ and

$$|\gamma_0| = \left|0.0538 + 0.0981(z_1 + z_1^{-1})\right| < 1, \ \forall |z_1| = 1,$$

which is equivalent with (see Remark 2)

$$|0.0538 + 2 \cdot 0.0981 \cos\omega_1| < 1, \ \forall \omega \in \mathbb{R}.$$

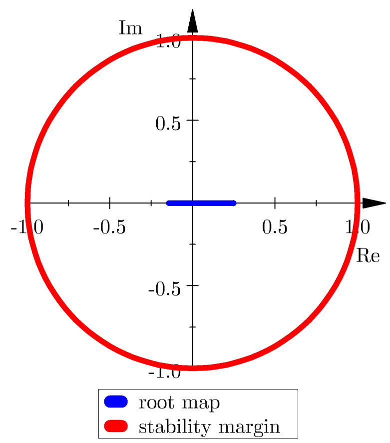

Fig. 2. Root map corresponding to (9) with $c_0 = 0.25$, $c_1 = 0.25$

Obviously, this inequality holds, thus the closed-loop system is stable.

Method described in Sec. 2.3 gives the Schur-Cohn matrix (in this case of order 1)

$$D_\chi(z_1) = 0.98 - 0.011\left(z_1 + z_1^{-1}\right) - 0.0096\left(z_1^2 + z_1^{-2}\right).$$

Choose real constant ε, for example $\varepsilon = 0.1$, and check whether $D_\chi(z_1) - \varepsilon$ has spectral factor. Generate the corresponding Toeplitz's matrix and make sure that it is positive definite. So, spectral factor exists, $D_\chi(z_1) > 0$ for all $|z_1| = 1$ holds, the closed-loop system is stable.

Using the method of Sec. 3.2, we have $D_\chi(1) = 0.9388 > 0$ and after the substitution we get

$$R(\zeta_1) = 0.98 - 0.11\,\zeta_1 - 0.0096\,(\zeta_1^2 - 2),$$

whose minimum with constraint $-2 \le \zeta_1 \le 2$ is $\mu = 0.9388$, so, $\mu > 0$ holds, the closed-loop system is stable.

Finally, use the method of Sec. 2.4. Kharitonov's polynomial corresponding to (10) is

$$z + [-0.25; 0.1424].$$

All the polynomials are stable, so the closed-loop system is also.

4.2 *A deformable mirror*

As the second example consider a deformable mirror with array of sensors and actuators, see e. g. Augusta and Hurák (2006), Cichy et al. (2008), described by the transfer function

$$P = \frac{b(z, z_1, z_1^{-1}, z_2, z_2^{-1})}{a(z, z_1, z_1^{-1}, z_2, z_2^{-1})} \tag{11}$$

with

$$b = 0.00000001\,z,$$
$$a = 2700\,z^2 - 5254.4\,z + 2700 + 1.73\,z\left({z_1}^2 + {z_1}^{-2}\right) + 15.6\,z\left({z_2}^2 + {z_2}^{-2}\right) - 45\,z\left(z_1\,z_2 + z_1^{-1}\,z_2 + z_1\,z_2^{-1} + z_1^{-1}\,z_2^{-1}\right).$$

In Fig. 3 one can see that root map lies on the unit circle and hence the transfer function (11) is not stable.

Let us now use the Schur-Cohn test as was described in Sec. 2.2. Since $a = \tilde{a}$, $F_0 = 1$, then $F_1 = 0$. So, we have

$$|\gamma_0| = 1 \not< 1$$

which indicates that the polynomial a is not stable.

Using the Schur-Cohn matrix test, described in Subsec. 2.3, we get the Schur-Cohn matrix

$$D_a(\mathbf{z}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

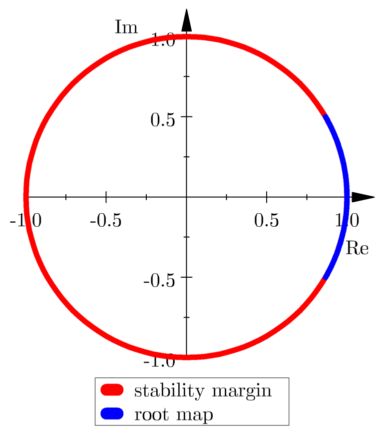

Fig. 3. Root map corresponding to system (11) with z_2 fixed to 1

and obviously $D_a(1) \not> 0$. The system is not stable.

Let us now consider the system (11) with a controller C and control scheme of Fig. 1. Suppose the controller has a form

$$C(z, z_1, z_2) = c_0 + z\Big(c_{00} + c_{10}(z_1 + z_1^{-1}) + c_{01}(z_2 + z_2^{-1}) + c_{11}(z_1\,z_2 + z_1^{-1}\,z_2 + z_1\,z_2^{-1} + z_1^{-1}\,z_2^{-1})\Big) \tag{12}$$

with

$$c_0 = -2700, \quad c_{00} = 5255,$$
$$c_{10} = -1.7, \quad c_{01} = -15, \quad c_{11} = 45.$$

The closed-loop system characteristic polynomial is

$$\chi = 2700\,z^2 + 0.6\,z + 1.7\,z\,(z_1 + z_1^{-1}) - 15\,z\,(z_2 + z_2^{-1}) + 1.73\,z\,(z_1^2 + z_1^{-2}) + 15.6\,z\,(z_2^2 + z_2^{-2}) \tag{13}$$

whose stability will be analysed.

The polynomial $\chi[z_1, z_2](z)$ has roots $\rho_1 = 0$ and $\rho_2 = -2\cdot 10^{-4} - 6\cdot 10^{-4}\,(z_1^2 + z_1^{-2}) - 5.7\cdot 10^{-3}\,(z_2^2 + z_2^{-2}) + 6\cdot 10^{-4}\,(z_1^2 + z_1^{-2}) + 5\cdot 10^{-3}\,(z_2^2 + z_2^{-2})$. Examples of the root map corresponding to ρ_2 for fixed values $z_2 = 1$ and $z_2 = -1$ are in Fig. 4. The system is stable.

The Schur-Cohn test gives

$$\begin{aligned} |\gamma_0| &= 0, \\ |\gamma_1| &= |-2\cdot 10^{-4} - 6\cdot 10^{-4}\,(z_1^2 + z_1^{-2}) \\ &\quad - 5.7,\cdot 10^{-3}\,(z_2^2 + z_2^{-2}) \\ &\quad + 6\cdot 10^{-4}\,(z_1^2 + z_1^{-2}) \\ &\quad + 5\cdot 10^{-3}\,(z_2^2 + z_2^{-2})|. \end{aligned} \tag{14}$$

$|\gamma_0| < 1$. Substitution $e^{j\omega_i}$ for z_i into (14) and manipulation give

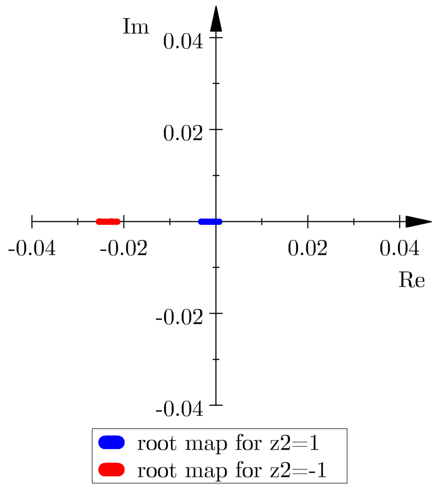

Fig. 4. Root map corresponding to system (13) with z_2 fixed to 1 and -1

$$|\gamma_1| = |-2\cdot 10^{-4} - 6\cdot 10^{-4}\cdot 2\,\cos(2\,\omega_1) - 5.7,\cdot 10^{-3}\cdot 2\,\cos(2\,\omega_2) + 6\cdot 10^{-4}\cdot 2\,\cos\omega_1 + 5\cdot 10^{-3}\cdot 2\,\cos\omega_2|$$

and obviously $|\gamma_1| < 1$. The system is stable.

The Schur-Cohn matrix D_χ reads

$$D_\chi(\mathbf{z}) = \begin{pmatrix} 7.29\cdot 10^6 & d_{12} \\ d_{21} & 7.29\cdot 10^6 \end{pmatrix},$$

where $d_{12} = d_{21} = 1620 - 4590(z_1 + z_1^{-1}) - 40500(z_2 + z_2^{-1}) + 4671(z_1^2 + z_1^{-2}) + 42120(z_2^2 + z_2^{-2})$. $D_\chi(\mathbf{1}) \succ 0$ and one can make sure that $\det D_\chi$ is positive polynomial on the unit circle with minimum equal to $5.3\cdot 10^{13}$. The result is the system is stable.

Finally, let us show use of Kharitonov's theorem to analyse stability of system with distributed parameters. Consider again the system (11) with the denominator polynomial

$$a = 2700\,z^2 - 5254.4\,z + 2700 + 1.73\,z\left({z_1}^2 + {z_1}^{-2}\right) + 15.6\,z\left({z_2}^2 + {z_2}^{-2}\right) - 45\,z\left(z_1\,z_2 + z_1^{-1}\,z_2 + z_1\,z_2^{-1} + z_1^{-1}\,z_2^{-1}\right).$$

Procedure described in Sec. 2.4 gives the following interval polynomial

$$\hat{a} = 2700\,z^2 - [5039.74; 5469.06]\,z + 2700. \quad (15)$$

For stability of a, stability of $\hat{a}$ is sufficient and in general not necessary condition. The interval polynomial $\hat{a}$ is stable if and only if corner polynomials

$$2700\,z^2 - 5039.74\,z + 2700$$
$$2700\,z^2 - 5469.06\,z + 2700$$

are stable, see Kraus et al. (1987). These polynomials are not stable. In this case of such the primitive example, where coefficients of (15) are not linked, we can say the polynomial (11) is not stable. In the case, where coefficients are linked, we can say nothing about stability of system and test which will give sufficient and necessary conditions has to be performed.

Consider now the closed loop system with characteristic polynomial (13)

$$\chi = 2700\,z^2 + 0.6\,z + 1.7\,z\,(z_1 + z_1^{-1}) - 15\,z\,(z_2 + z_2^{-1}) + 1.73\,z\,(z_1^2 + z_1^{-2}) + 15.6\,z\,(z_2^2 + z_2^{-2}).$$

The corresponding interval polynomial is

$$\hat{\chi} = 2700\,z^2 + [-67.46; 68.66]\,z,$$

which is stable if and only if corner polynomials

$$2700\,z^2 - 67.46\,z$$
$$2700\,z^2 + 68.66\,z$$

are stable. Both the above polynomials are stable, so (13) is also.

5. CONCLUSIONS

Stability tests for spatially distributed systems described by a fraction of multivariate polynomials were presented in this paper. Two approaches to Schur-Cohn criterion were described and extended to multivariate polynomials corresponding to sequences with half-plane support. The method based on Schur-Cohn matrix was formulated as a problem of positivity of polynomial matrix on the unit circle, which was solved by two various techniques. The first one uses equivalence of univariate polynomial positivity with existence of its spectral factor and can be use for 2-D systems, the second one is general and is based on minimisation with constraint.

Use of Kharitonov's theorem to decide on stability of spatially distributed system was also mentioned. This method gives generally only sufficient condition and can serve as the first step in finding out stability of multivariate polynomial. If the result is "stable", the system is really stable, but if the result is "not stable", another stability test which is able to give sufficient and necessary conditions has to be performed.

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