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CONSTRAINED NMPC USING POLYNOMIAL CHAOS THEORY

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Abstract: Establishing an accurate model of a multivariable nonlinear process with uncertain parameters can be difficult. Application of control methods based on nonlinear optimization may result in sub-optimal performance due to changes in the parameters. This paper presents a new control method to handle parametric uncertainty through incorporation of a Polynomial Chaos Theory (PCT) model used in a constrained Nonlinear Model Predictive Control (NMPC) formulation. Uncertain parameters are treated as random variables with a uniform distribution. PCT expresses the entire uncertain process by a complete and orthogonal Legendre polynomial basis in terms of random variables where expanded process outputs are determined by applying Galerkin projection onto the polynomial basis. NMPC formulation has the ability to apply hard input and soft output constraints to maintain the process within specified bounds. It is shown that the proposed formulation can be applied with an adequate tuning to minimize the effect of parametric uncertainty on the process outputs.

Keywords: Model Predictive Control, Polynomial Chaos Theory, Parametric Uncertainty

1. INTRODUCTION

Parametric uncertainty affects the quality of a process model and as a result brings in significant challenges for process control engineering, design and analysis. Different methods have been used to better analyze and simulate the uncertain systems: Monte Carlo and other statistical methods , Taylor expansion of the random variables, worst cases scenarios and qualitative analysis of prediction algorithms M. Papagelis (2005). While some of these methods are expensive and require parallel simulations to obtain the full statistics after each time step Lovett (2004), and others are related to artificial intelligence and the field of decision making not currently applicable for large-scale engineering applications, Polynomial Chaos Theory (PCT) is a deterministic method that is capable of calculating the entire statistics of each uncertain variable during only one simulation. PCT analysis that includes polynomial expansion of the uncertain variables results in a multivariable system while the statistical information required for reconstruction of the original variables is stored in the form of coefficients in the basis spanned by the polynomials.

Model Predictive Control (MPC) refers to a class of control algorithms in which a dynamic model of the plant is used to predict and optimize the future behavior of the process Garcia et al. (1989); Meadows and Rawlilngs (1997). At each control interval, the MPC algorithm computes a sequence of the manipulated variables to optimize the future behavior of the plant. MPC has been used extensively for control of refinery operations since MPC can accommodate multivariable systems, actuator constraints, and economic objectives. The original linear MPC method has been extended to include control of nonlinear dynamic systems by a variety of authors Biegler and Rawlings (1991); Ricker and Lee (1995); Henson and Seborg (1997); Ettedgui et al. (1997); Marco et al. (1997); Allgwer et al. (1999); Qin and Badgwell (1999); Biegler (1998). Use of more accurate nonlinear process models potentially results in improved controller performance but also requires solution of a more difficult nonlinear optimization problem. Most commercially available MPC technologies are based on a linear process model. For processes that are highly nonlinear, the performance of MPC based on a linear model can be poor. This led to the development of Nonlinear Model Predictive Control (NMPC) methods Allgwer and Zheng (1993); Henson (1998); Biegler and Rawlings (1991); Biegler (1998); Marco et al. (1997); R. S. Parker and E. P. Gatzke and R. Mahadevan and E. S. Meadows and F. J. Doyle III (2001).

Many of the current NMPC schemes are based on first principles physical models of the process. However, in many cases such models are difficult to obtain, time-consuming and often not available. Process simulators can be used to obtain a nonlinear empirical mathematical model which is identified from input-output data Qin and Badgwell (1999). While NMPC offers potential for improved process operation, the method also faces practical issues that are considerably more challenging than those associated with linear MPC. In particular, the problems associated with the nonlinear optimization routine that must be solved online at each sample period to generate the optimal control sequence. Guaranteed closed-loop stability of nonlinear systems using MPC based methods generally use a terminal state constraint Muske and Rawlings (1994); Sckokaert et al. (1999); Mayne et al. (2000) or some sort of backup control system that monitors convergence Mhaskar and an P. D. Christofides (2005). The nonlinearity of a refining process and multivariable interacting nature of such systems makes this class of process attractive to nonlinear MPC methods Simminger et al. (1991); Skogestad and Postlethwaite (1996).

When implementing control on real multivariable chemical or petrochemical processes such as distillation or separation operations, it is essential to ensure that the process remain within established safety limits and that each product meet certain quality constraints and specifications. For control purposes, all safety constraints and product quality specifications provide a set of control objectives that must be satisfied. However, in situations where the process is characterized by limited degrees of freedom (due to an input actuator saturation, nonsquare process with limited inputs) it typically becomes impossible for a controller to meet all control objectives. In these types of cases it is practically impossible for a controller to impose hard constraints on the process outputs. Direct incorporation of hard output constraints would generally lead to infeasibility in the optimization problem.

Since constrained MPC requires the solution of an optimization problem at each time step, the feasibility of that problem should be ensured. Use of a terminal state constraint to guarantee closed-loop stability can cause the nonlinear MPC optimization problem to become infeasible. If the online optimization problem is not feasible, then some constraints would have to be relaxed and the problem would be resolved. Determining the constraints one must relax in order to get a feasible problem with optimal deterioration of the objective function could be extremely difficult. A possible remedy to the problem is to consider prioritized soft constraints on process outputs by including a penalty term in the objective function.

The paper is organized as follows: first, Polynomial Chaos Theory is presented in Section 2. The proposed controller formulation is presented in Section 3 along with an explanation of a methodology for handling output constraints. The case-study used in this paper is presented and discussed in Section 4. This work uses a two-tank model as a nonlinear case-study. Open-loop and closed-loop results are presented in Sections 5 and 6, respectfully, and conclusions are drawn in Section 7.

2. POLYNOMIAL CHAOS THEORY

Polynomial Chaos Theory (PCT) was first introduced in 1938 by an American mathematician Norbert Wiener. Wiener used Hermite polynomials to expand continuous uncertain variables into a stochastic space and represent the uncertainty in the form of probability distribution function (PDF). The approach used by Wiener was later broadened for the entire Askey scheme of orthogonal polynomials and was renamed Wiener-Askey Polynomial Chaos Xiu and Karniadakis (2002). Any continuous uncertain variable $X(\omega)$ can be generally described using Polynomial Chaos method as follows:

$$X(\omega) = \sum_{i=0}^{\infty} x_i \, \phi_i(\xi(\omega)) \tag{1}$$

where ξ represents random variables in terms of ω with the type of probability distribution function suitable for the chosen polynomial basis ϕ_i, and x_i are the coefficients of expansion for this uncertain variable. The infinite dimension of the polynomial space given in Equation 1 must be replaced for computational use by a finite dimension P:

$$X(\omega) = \sum_{i=0}^{P} x_i \, \phi_i(\xi\,(\omega)) \tag{2}$$

Note that P equals the number of terms in the expansion starting from 0. In general, the number of terms P needed to describe each uncertain variable in a PCT expanded model can be obtained using

$$P = \left(\frac{(n+k)!}{n!\,k!} - 1 \right) \tag{3}$$

where n is the number of random variables ξ_i and k is the maximum order of the polynomial basis to be used. Two cases are analyzed in this paper and appear in the following subsections: a case of only one uncertain parameter and a case of two uncertain parameters, i.e. $n = 1$ and $n = 2$, respectively.

2.1 *Uncertainty in One Parameter*

For a process in which only one parameter is uncertain, Equation 3 becomes for $n = 1$:

$$P = \left(\frac{(1+k)!}{k!} - 1 \right) \tag{4}$$

which corresponds to $P = k$. This means that the number of terms needed to deterministically represent the stochastic process equals the order of the polynomial expansion. Assuming uniform distribution can choose Legendre polynomials to be used for PCT expansion. The interval of orthogonality for Legendre Polynomials is [-1, 1] and the weighting factor is 1. This translates into two inner product definitions and orthogonality conditions for Legendre polynomials expressed in terms of the 2^{nd}-order Kronecker delta function δ_{mn} and a 3^{rd}-order tensor C_{ijk}, respectfully:

$$\langle \phi_i \phi_j \rangle \equiv \left\langle \phi_i^2 \right\rangle \delta_{ij} = \int_{-1}^{1} \phi_i(\xi)\phi_j(\xi)\,1\,d\xi \tag{5}$$

$$\langle \phi_i \phi_j \phi_k \rangle \equiv \left\langle \phi_k^2 \right\rangle C_{ijk} = \int_{-1}^{1} \phi_i(\xi)\phi_j(\xi)\phi_k(\xi)\,1\,d\xi \tag{6}$$

The orthogonality normalization factors $\left\langle \phi_i^2 \right\rangle$ and $\left\langle \phi_k^2 \right\rangle$ that appear in Equations 5 and 6 can be found in Table 1 up to order 2. The first two terms (starting from 0) in the case of a second-order Legendre polynomial basis are obtained from Table 1: $\phi_0\,(\xi) = 1$, $\phi_1\,(\xi) = \xi$, and $\phi_2(\xi) = 0.5\left(3\,\xi^2 - 1\right)$, so that the full PCT expansion of a variable X in the process model is expressed in the case of only one uncertain parameter and the second-order Legendre polynomials by:

order k	$\phi_k(\xi)$	$\int_{-1}^{1} \phi_k^2(\xi)\,d\xi$
0	1	2
1	ξ	2/3
2	$0.5\,(3\,\xi^2 - 1)$	2/5

Table 1. Legendre polynomial terms up to order 2 and orthogonality normalization factors in the case of only one uncertain parameter.

k →	0			1			2		
i →	0	1	2	0	1	2	0	1	2
j=0	1	0	0	0	1	0	0	0	1
j=1	0	1/3	0	1	0	2/5	0	2/3	0
j=2	0	0	1/5	0	2/5	0	1	0	2/7

Table 2. Elements of the 3^{rd}-order tensor C_{ijk}

$$\begin{aligned} X &= \sum_{i=0}^{2} x_i \, \phi_i(\xi) \\ &= x_0 + x_1\,\xi + 0.5\,x_2\,\left(3\,\xi^2 - 1\right) \end{aligned} \tag{7}$$

Once all the variables in the system are expanded according to Equation 2, the resulting expressions are substituted into the governing model equation to form a PCT expanded model equation. The latter may in turn be discretized using Galerkin projection B. Cockburn (2000); Rice and Do (1995) onto the polynomial chaos basis in Equation 2 and then expressed in terms of the coefficients x_i, the Kronecker delta function δ_{ij} and the 3^{rd}-order tensor C_{ijk}.

The terms $\langle \phi_i \phi_j \phi_k \rangle$ or, alternatively, C_{ijk} can be calculated up to order 2 (total of $3^3 = 27$ terms) using Legendre polynomials and normalization factors from Table 1 above. The terms of a tensor C_{ijk} are presented for each combination of i , j , and k in Table 2 Smith (2007).

Using the described procedure and the data presented in the tables above, the Polynomial Chaos Theory analysis results in a new expanded deterministic model of a higher order. In fact, if the original governing model consists of n differential equations, then the expanded model in the case of one uncertain parameter will consist of $n\,(k+1)$ equations, where k is the order of the PCT expansion. The resulting PCT expanded model does not include the random variables, and if presented in a state-space model, the states of the new model are the expansion coefficients x_i from Equation 2.

First order $k = 1$	Second order $k = 2$
$\phi_0 = 1$	$\phi_3 = \xi_1\xi_2$
$\phi_1 = \xi_1$	$\phi_4 = 0.5\,(3\,\xi_1^2 - 1)$
$\phi_2 = \xi_2$	$\phi_5 = 0.5\,(3\,\xi_2^2 - 1)$

Table 3. Legendre polynomial terms up to order 2 and orthogonality normalization factors in the case of two uncertain parameters.

2.2 *Uncertainty in Two Parameters*

For a process in which two parameters are uncertain, Equation 3 becomes for $n = 2$:

$$P = \left(\frac{(2+k)!}{2!\,k!} - 1\right) \tag{8}$$

Using Equation 8 for a second-order polynomial basis, i.e. $k = 2$, it is established that $P = 5$, which means that five terms are required in the PCT polynomial expansion in the form of Equation 2. Note that two random variables ξ_1 and ξ_2 now need to be used to express the terms of Legendre polynomials. The terms up to order 2 appear in Table 3. The detailed PCT analysis of this case does not appear in this paper. However, the analysis is very similar to the one presented in this work.

3. NMPC HANDLING SOFT CONSTRAINTS

For a continuous nonlinear state-space model of the form

$$\begin{aligned} \frac{dx}{dt}(t) &= f\left(t, x\left(t\right), u\left(t\right)\right) \\ y\left(t\right) &= h\left(t, x\left(t\right), u\left(t\right)\right) \end{aligned} \tag{9}$$

a general nonlinear discrete time dynamic model with M past input terms, n_u inputs, n_y outputs, move horizon of m, and prediction horizon of p, is formulated according to:

$$y_j(k+M) = \sum_{i=1}^{n_u}\sum_{l=k}^{k+M-1}\left[\alpha_{j,i}(l)\,g\left(u_i(l)\right)\right] \tag{10}$$

along with a constant output disturbance term as:

$$y_j(k)|_{k\in[1,\,p]} = y_j^{Model} + d_j \tag{11}$$

In equation 10, coefficients $\alpha_{j,i}\left(l\right)$ relate output j to a general nonlinear input term $g\left(u_i(l)\right)$ at each time l. In equation 11, y_j^{Model} is the model of output j using the discrete representation in the form given in Equation 10, $y_j(k)$ is a predicted value of output j at time k, and the disturbance update d_j is defined as:

$$d_j = y_j^{Model}(0) - y_j^{Meas}(0) \tag{12}$$

where for each output j, $y_j^{Model}(k)$ is the model value at the current time $k = 0$ and $y_j^{Meas}(k)$ is the process measurement at the current time $k = 0$. In this model, values for u_i before time $k = 0$ are known and values for times greater than $m-1$ are fixed to $u(k+m-1)$. This formulation chooses a sequence of input moves over the move horizon (m) that minimizes a 2-norm cost function. A 2-norm is used in the MPC objective function in this work to avoid performance issues associated with the 1-norm formulations Rao and Rawlings (2000). The 2-norm objective function with soft constraints takes the form:

$$\begin{aligned} \phi = &\sum_{j=1}^{n_y}\sum_{k=1}^{p}\Gamma_{y,j}\,\left(e_j(k)\right)^2 + \sum_{i=1}^{n_u}\sum_{l=1}^{m}\Gamma_{u,i}\,\left(\Delta u_i(l)\right)^2 \\ &+\sum_{j=1}^{n_y}\sum_{k=1}^{p}\Gamma_{y_{soft},j}\,\left(s_j(k)\right)^2 \end{aligned} \tag{13}$$

where $e_j(k)$ and $s_j(k)$ are the values of error predicted for the k^{th} time step into the future for each output j. The error (e) is defined as

$$e_j(k)|_{k\in[1,\,p]} = y_{p,j}(k) - y_{sp,\,j}(k) \tag{14}$$

where $y_{sp,\,j}(k)$ is the known setpoint value of output j at time k and $y_{p,\,j}(k)$ is the predicted value of output j at time k, updated based upon process model mismatch at the current time. The term Δu_i defines changes in input i according to

$$\begin{aligned} \Delta u_i(k)|_{k\in[1,\,m]} = &\,u_i(M+k) \\ &-u_i(M+k-1) \end{aligned} \tag{15}$$

The soft constraint violation (s) is defined for those values of output j that are outside the range$\left[y_{soft}^{l}(k), \cdots y_{soft}^{u}(k)\right]$. For model predictions above the upper soft constraint limit, the soft constraint violation is defined as:

$$s_j(k)|_{k\in[1,\,p]} = y_{p,j}(k) - y_{soft,\,j}^{u}(k) \tag{16}$$

For violation below the lower soft constraint limit, this violation is defined as:

$$s_j(k)|_{k\in[1,\,p]} = y_{soft,\,j}^{l}(k) - y_{p,j}(k) \tag{17}$$

The soft constraint violation is zero otherwise. This allows for violation of output constraints without making the controller optimization problem infeasible. $\Gamma_{y,j}$, $\Gamma_{u,i}$ and $\Gamma_{y_{soft},j}$ are weighting factors used to define the relative importance of each objective function term in Equation 13. However, penalty values must be tuned for this process using the weight in the Γ_{soft} matrix. The term $\Gamma_{y_{soft},j}$ is an penalty on the output error that is applied depending on soft constraints on that output. Note that the value of $\Gamma_{y_{soft},j}$ is at least an order of magnitude larger than the maximum value of $\Gamma_{y,j}$. This is done to ensure that soft constraint violations are minimized as much as possible.

In some cases, there may be multiple output constraints. For example, a process may make multiple product types that are defined by the measured product quality. High profit products must meet stringent quality limits while lower quality products may be sold at a lower cost. To implement this approach, based on deviation from the setpoint error in Equation 14, multiple layers of soft constraints with different penalties can be implemented. In this formulation, the value of the soft penalty $\Gamma_{y_{soft},j}$ in the objective function, Equation 13, increases by order of magnitude with each additional layer, starting from a tight range (for errors less than or equal to some small value ϵ) and ending with a broader loose range (for errors greater than 1.5ϵ, for example). The value of the quality range ϵ as well as a penalty used for each layer of constraints can be adjusted to process needs depending on the output and the sign of the error in Equation 14, i.e. the constraint layers boundaries are not necessarily symmetrical around the setpoint.

Combining Equations 11, 12, and 14-17 results in a single objective function (Equation 13) that depends only on the input values. The resulting optimization problem becomes:

$$\begin{gathered} \min \phi \\ u^l \leq u \leq u^u \end{gathered} \tag{18}$$

The inclusion of soft constraints that are only active in portions of the parameter space make the objective function nonsmooth. The convexity of the objective function ϕ could be examined in detail.

Optimization is implemented using *fminsearch* - multidimensional nonlinear minimization - in Matlab. In MPC formulations, the prediction horizon (p) can be chosen as a large value to promote stability. Stability can also be ensured through the use of a hard constraint which drives the terminal state error to zero. This theoretical guarantee for nominal stability fails in cases where an unreachable setpoint is provided, as the optimization problem is infeasible Long et al. (2006). In such cases, a soft constraint could be used to drive the system to a stable operating point when possible.

4. CASE STUDY: TWO TANKS IN SERIES

The case study analyzed in this paper is a simple two tank model described in Figure 1. The constant cross-sectional tank areas are A_1 and A_2, and the liquid heights are h_1 and h_2, respectfully. There are two valves at the outlet of each tank whose coefficients are k_2 and k_3. The flowrate into

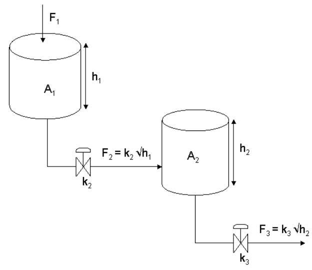

Fig. 1. Flowchart of two connected tanks with liquid levels h_1 and h_2, cross-sectional areas A_1 and A_2.

the first tank is $F_1(t)$, the flowrates from the tanks are proportional to the valve coefficients and the square root terms of the liquid levels through:

$$\begin{aligned} F_2(t) &= k_2 \sqrt{h_1(t)} \\ F_3(t) &= k_3 \sqrt{h_2(t)} \end{aligned} \tag{19}$$

The material balance around the system results in the following mathematical model:

$$\begin{aligned} \frac{dh_1(t)}{dt} &= \frac{1}{A_1}(F_1(t) - k_2 \sqrt{h_1(t)}) \\ \frac{dh_2(t)}{dt} &= \frac{1}{A_2}(k_2 \sqrt{h_1(t)} - k_3 \sqrt{h_2(t)}) \end{aligned} \tag{20}$$

It is worth noting at this point that in most complex chemical processes it is impossible for a mathematical model to fully represent all the aspects of the ongoing process operation. However, the nonlinear model given in Equation 20 is considered the most suitable representation of the two-tank model. Note that time dependency of all the variables was omitted in the model equation.

In order to effectively analyze the nonlinearity of the system, the square root terms $\sqrt{h_i}$ can be approximated using the Taylor series expansion in the neighborhood of the points h_i^0:

$$\begin{aligned} f(h_i) = f(h_i^0) + f'(h_i^0)(h_i - h_i^0) + \frac{f''(h_i^0)}{2}(h_i - h_i^0)^2 \\ + \dots \frac{f^{(n)}(h_i^0)}{n!}(h_i - h_i^0)^n \end{aligned} \tag{21}$$

where $f(h_i) = \sqrt{h_i}$.

4.1 *Uncertainty in One Parameter k_2*

4.1.1. *First-Order Taylor Approximation*

Using the first order Taylor series expansion given in Equation 21, the nonlinear square term $\sqrt{h_i(t)}$ for any i reduces to:

$$\sqrt{h_i(t)} = \sqrt{h_i^0} + \frac{1}{2\sqrt{h_i^0}}(h_i(t) - h_i^0) \tag{22}$$

If h_i^0 is known, one can substitute Equation 22 into the model in Equation 20 so that the modified model becomes:

$$\frac{dh_1(t)}{dt} = \frac{1}{A_1}\left[F_1(t) - \frac{1}{2}k_2\sqrt{h_1^0} - \frac{k_2}{2\sqrt{h_1^0}}h_1(t)\right]$$
$$\frac{dh_2(t)}{dt} = \frac{1}{A_2}\left[\frac{1}{2}k_2\sqrt{h_1^0} + \frac{k_2}{2\sqrt{h_1^0}}h_1(t) - \frac{1}{2}k_3\sqrt{h_2^0} - \frac{k_3}{2\sqrt{h_2^0}}h_2(t)\right] \tag{23}$$

Since the first valve coefficient k_2 is the only uncertain variable, one can apply polynomial chaos expansion to the state variables h_1 and h_2, and the only uncertain parameter k_2 to get:

$$k_2(\xi) = \sum_{i=0}^{P} k_{2,i}\,\phi_i(\xi)$$
$$h_1(t,\xi) = \sum_{i=0}^{P} h_{1,i}(t)\,\phi_i(\xi) \tag{24}$$
$$h_2(t,\xi) = \sum_{i=0}^{P} h_{2,i}(t)\,\phi_i(\xi)$$

In this PCT expansion, $\phi_i(\xi)$ (where ξ is a random variable with uniform distribution) can be chosen as Legendre polynomials for which the interval of orthogonality is [-1, 1] and the weighting function is simply 1, so that the first three Legendre polynomial terms (for P = 2) in the case of only one uncertain parameter as given in Table 1 are: $\phi_0(\xi) = 1$, $\phi_1(\xi) = \xi$, and $\phi_2(\xi) = \frac{1}{2}(3\,\xi^2 - 1)$. Now one can insert Equation 24 into Equation 23 to obtain:

$$\sum_{i=0}^{P}\frac{dh_{1,i}(t,\xi)}{dt}\phi_i(\xi) = \frac{1}{A_1}\left[F_1(t) - \frac{1}{2}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\,\phi_i(\xi)\right] - \frac{1}{A_1}\left[\frac{1}{2\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,\phi_i(\xi)\,\phi_j(\xi)\right]$$
$$\sum_{i=0}^{P}\frac{dh_{2,i}(t,\xi)}{dt}\phi_i(\xi) = \frac{1}{A_2}\left[\frac{1}{2}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\,\phi_i(\xi)\right] + \frac{1}{A_2}\left[\frac{1}{2\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,\phi_i(\xi)\,\phi_j(\xi)\right] - \frac{1}{A_2}\left[\frac{1}{2\sqrt{h_2^0}}k_3\sum_{j=0}^{P}h_{2,j}(t)\,\phi_j(\xi) - \frac{1}{2}k_3\sqrt{h_2^0}\right] \tag{25}$$

Applying the orthogonality condition given in Equation 5 and using a tensor notation from Equation 6 it is possible to discretize the PCT expanded model in Equation 25 using Galerkin projection onto the polynomial chaos basis in Equation 24 to get:

$$(\forall k \in \{0,\ldots,P\})\quad \frac{dh_{1,k}(t)}{dt} = \frac{\int_{-1}^{1}\left\{\frac{1}{A_1}\left[F_1(t) - \frac{1}{2}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\,\phi_i(\xi)\right]\right\}\phi_k\,d\xi}{\int_{-1}^{1}\phi_k^2\,d\xi} + \frac{\int_{-1}^{1}\left\{\frac{1}{A_1}\left[-\frac{1}{2\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,\phi_i(\xi)\,\phi_j(\xi)\right]\right\}\phi_k\,d\xi}{\int_{-1}^{1}\phi_k^2\,d\xi}$$
$$\frac{dh_{2,k}(t)}{dt} = \frac{\int_{-1}^{1}\left\{\frac{1}{A_2}\left[\frac{1}{2}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\,\phi_i(\xi) - \frac{1}{2\sqrt{h_2^0}}k_3\sum_{j=0}^{P}h_{2,j}(t)\,\phi_j(\xi)\right]\right\}\phi_k\,d\xi}{\int_{-1}^{1}\phi_k^2\,d\xi} + \frac{\int_{-1}^{1}\left\{\frac{1}{A_2}\left[\frac{1}{2\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,\phi_i(\xi)\,\phi_j(\xi) - \frac{1}{2}k_3\sqrt{h_2^0}\right]\right\}\phi_k\,d\xi}{\int_{-1}^{1}\phi_k^2\,d\xi} \tag{26}$$

This equation represents the PCT expanded two tanks model, where instead of two original model equations the expanded model now consists of $2(k+1)$ equations. It can be modified and rewritten in terms of a 3^{rd}-order tensor C_{ijk} and the Kronecker delta δ_{mn} using Equations 6 and 5, respectively:

$$\frac{dh_{1,k}(t)}{dt} = \frac{1}{A_1}\left[-\frac{1}{2\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,C_{ijk}\right] + \frac{1}{A_1}\left[\frac{\int_{-1}^{1}F_1(t)\,\phi_k d\xi}{\int_{-1}^{1}\phi_k^2 d\xi} - \frac{1}{2}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\delta_{ik}\right]$$
$$\frac{dh_{2,k}(t)}{dt} = \frac{1}{A_2}\left[\frac{1}{2\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,C_{ijk}\right] + \frac{1}{A_2}\left[\frac{1}{2}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\delta_{ik} - \frac{1}{2\sqrt{h_2^0}}k_3\sum_{j=0}^{P}h_{2,j}(t)\,\delta_{jk}\right] + \frac{1}{A_2}\left[-\frac{\int_{-1}^{1}\frac{1}{2}k_3\sqrt{h_2^0}\phi_k\,d\xi}{\int_{-1}^{1}\phi_k^2 d\xi}\right] \tag{27}$$

A denominator $\int_{-1}^{1}\phi_k^2\,d\xi$ in Equations 26 and 27 that accounts for orthogonality of Galerkin projection is given for k up to order 2 in Table 1. The terms $\langle\phi_i\phi_j\phi_k\rangle$ or, alternatively, C_{ijk} can be calculated using the inner product definition in

Equation 6. These terms are summarized in Table 2.

In the first part of the resulting PCT expanded model (Equation 27) a term $\int_{-1}^{1} F_1(t)\,\phi_k d\xi$ contributes only when $k = 0$ due to the properties of Legendre polynomials. For this simplest case of zero-order, the resulting model consisting of two differential equations is:

$$\begin{aligned}\frac{dh_{1,0}(t)}{dt} &= \frac{1}{A_1}\left[F_1(t) - \frac{1}{2}k_{2,0}\sqrt{h_1^0}\right] \\ &-\frac{1}{A_1}\left[\frac{1}{2\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t)\right] \\ \frac{dh_{2,0}(t)}{dt} &= \frac{1}{A_2}\left[\frac{1}{2}k_{2,0}\sqrt{h_1^0} - \frac{1}{2}k_3\sqrt{h_2^0}\right] \\ +\frac{1}{A_2}&\left[\frac{1}{2\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t) - \frac{1}{2\sqrt{h_2^0}}\,k_3 h_{2,0}(t)\right]\end{aligned} \tag{28}$$

For this zero-order model, $C_{000} = 1$ was used. The resulting first-order PCT expanded model consists of four differential equations with up to $P = 1$ terms in each. Using the values of C_{ijk} for $k = 0$ and $k = 1$, one can obtain from Equation 27:

$$\begin{aligned}\frac{dh_{1,0}(t)}{dt} &= \frac{1}{2A_1} \\ &\left[2F_1(t) - \frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t) - \frac{1}{3\sqrt{h_1^0}}\,k_{2,1}\,h_{1,1}(t) - \sqrt{h_1^0}\,k_{2,0}\right] \\ \frac{dh_{1,1}(t)}{dt} &= \frac{1}{2A_1} \\ &\left[-\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,1}(t) - \frac{1}{\sqrt{h_1^0}}\,k_{2,1}\,h_{1,0}(t) - \sqrt{h_1^0}\,k_{2,1}\right] \\ \frac{dh_{2,0}(t)}{dt} &= \frac{1}{2A_2} \\ &\left[\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t) + \frac{1}{3\sqrt{h_1^0}}\,k_{2,1}\,h_{1,1}(t)\right] \\ &+\frac{1}{2A_2}\left[-\frac{1}{\sqrt{h_2^0}}\,k_3\,h_{2,0}(t) - k_3\sqrt{h_2^0} + \sqrt{h_1^0}\,k_{2,0}\right] \\ \frac{dh_{2,1}(t)}{dt} &= \frac{1}{2A_2}\left[\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,1}(t) + \sqrt{h_1^0}\,k_{2,1}\right] \\ &+\frac{1}{2A_2}\left[\frac{1}{\sqrt{h_1^0}}\,k_{2,1}\,h_{1,0}(t) - \frac{1}{\sqrt{h_2^0}}\,k_3\,h_{2,1}(t)\right]\end{aligned} \tag{29}$$

Equation 29 can also be rewritten in a state-space form:

$$\begin{aligned}\frac{d\underline{x}_{PCT}(t)}{dt} &= A_{PCT}(t)\,\underline{x}_{PCT}(t) \\ &+B_{PCT}(t)\,\underline{u}(t) + \Gamma_{PCT}\end{aligned} \tag{30}$$

$$\underline{y}_{PCT}(t) = C_{PCT}(t)\,\underline{x}_{PCT}(t) + D_{PCT}(t)\,\underline{u}(t)$$

where $\underline{x}_{PCT}(t) = [h_{1,0}(t)\ h_{1,1}(t)\ h_{2,0}(t)\ h_{2,1}(t)]^T$ is a vector of expanded states, and $\underline{y}_{PCT}(t) = \underline{x}_{PCT}(t)$ is the output vector that includes all the expanded states.

Matrices A_{PCT}, B_{PCT}, C_{PCT}, D_{PCT} and Γ_{PCT} can then be identified as follows:

$$A_{PCT} = \begin{pmatrix} -\frac{1}{2}\frac{k_{2,0}}{A_1\sqrt{h_1^0}} & -\frac{1}{6}\frac{k_{2,1}}{A_1\sqrt{h_1^0}} & 0 & 0 \\ -\frac{1}{2}\frac{k_{2,1}}{A_1\sqrt{h_1^0}} & -\frac{1}{2}\frac{k_{2,0}}{A_1\sqrt{h_1^0}} & 0 & 0 \\ \frac{1}{2}\frac{k_{2,0}}{A_2\sqrt{h_1^0}} & \frac{1}{6}\frac{k_{2,1}}{A_2\sqrt{h_1^0}} & -\frac{1}{2}\frac{k_3}{A_2\sqrt{h_2^0}} & 0 \\ \frac{1}{2}\frac{k_{2,1}}{A_2\sqrt{h_1^0}} & \frac{1}{2}\frac{k_{2,0}}{A_2\sqrt{h_1^0}} & 0 & -\frac{1}{2}\frac{k_3}{A_2\sqrt{h_2^0}} \end{pmatrix} \tag{31}$$

$$B_{PCT} = \begin{pmatrix} \frac{1}{A_1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{32}$$

$$\Gamma_{PCT} = \begin{pmatrix} -\frac{1}{2A_1}\sqrt{h_1^0}\,k_{2,0} \\ -\frac{1}{2A_1}\sqrt{h_1^0}\,k_{2,1} \\ \frac{1}{2A_2}\sqrt{h_1^0}\,k_{2,0} - k_3\frac{1}{2A_2}\sqrt{h_2^0} \\ \frac{1}{2A_2}\sqrt{h_1^0}\,k_{2,1} \end{pmatrix} \tag{33}$$

$$C_{PCT} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \quad D_{PCT} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{34}$$

The second-order PCT expanded model consists of six differential equations with $P = 2$ terms in each.

$$\begin{aligned}\frac{dh_{1,0}(t)}{dt} &= \frac{1}{2A_1}\left[2F_1(t) - \frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t)\right] \\ &+\frac{1}{2A_1}\left[-\frac{1}{3\sqrt{h_1^0}}\,k_{2,1}\,h_{1,1}(t) - \sqrt{h_1^0}\,k_{2,0}\right] \\ \frac{dh_{1,1}(t)}{dt} &= \frac{1}{2A_1}\left[-\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,1}(t) - \sqrt{h_1^0}\,k_{2,1}\right] \\ &+\frac{1}{2A_1}\left[-\frac{1}{\sqrt{h_1^0}}\,k_{2,1}\,h_{1,0}(t) - \frac{2}{5\sqrt{h_1^0}}\,k_{2,1}\,h_{1,2}\right] \\ \frac{dh_{1,2}(t)}{dt} &= \frac{1}{2A_1}\left[-\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,2}(t) - \frac{2}{3\sqrt{h_1^0}}\,k_{2,1}\,h_{1,1}(t)\right] \\ \frac{dh_{2,0}(t)}{dt} &= \frac{1}{2A_2}\left[\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t) + k_{2,0}\sqrt{h_1^0}\right] \\ &+\frac{1}{2A_2}\left[-\frac{1}{\sqrt{h_2^0}}\,k_3 h_{2,0}(t) + \frac{1}{3\sqrt{h_1^0}}\,k_{2,1}\,h_{1,1}(t) - k_3\sqrt{h_2^0}\right] \\ \frac{dh_{2,1}(t)}{dt} &= \frac{1}{2A_2}\left[\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,1}(t) + \sqrt{h_1^0}\,k_{2,1}\,h_{1,0}(t)\right] \\ &+\frac{1}{2A_2}\left[+\frac{2}{5\sqrt{h_1^0}}\,k_{2,1}\,h_{1,2} - \frac{1}{\sqrt{h_2^0}}\,k_3 h_{2,1}(t) + k_{2,1}\sqrt{h_1^0}\right] \\ \frac{dh_{2,2}(t)}{dt} &= \frac{1}{2A_2}\left[\frac{1}{\sqrt{h_1^0}}\,k_{2,0}\,h_{1,2}(t)\right] \\ &+\frac{1}{2A_2}\left[\frac{2}{3\sqrt{h_1^0}}\,k_{2,1}\,h_{1,1} - \frac{1}{\sqrt{h_2^0}}\,k_3 h_{2,2}(t)\right]\end{aligned} \tag{35}$$

4.1.2. *Second-Order Taylor Approximation* Using the second order Taylor series expansion given in Equation 21, the nonlinear square term $\sqrt{h_i}$ for any i reduces to:

$$\sqrt{h_i} = \sqrt{h_i^0} + \frac{1}{2\sqrt{h_i^0}}(h_i - h_i^0) - \frac{1}{8\left(\sqrt{h_i^0}\right)^3}\left(h_i - h_i^0\right)^2 \quad (36)$$

If h_i^0 is known, can substitute Equation 36 into the model in Equation 20 so that the modified model consisting of two differential equations becomes:

$$\begin{aligned}\frac{dh_1(t)}{dt} &= \frac{k_2}{A_1}\left[\frac{F_1(t)}{k_2} - \frac{1}{2}\sqrt{h_1^0} + \frac{1}{8\left(\sqrt{h_1^0}\right)^3}\left(h_1^0\right)^2\right] \\ &+ \frac{k_2}{A_1}\left[-\left(\frac{1}{2\sqrt{h_1^0}} + \frac{h_1^0}{4\left(\sqrt{h_1^0}\right)^3}\right)h_1(t)\right] \\ &+ \frac{k_2}{A_1}\left[\frac{1}{8\left(\sqrt{h_1^0}\right)^3}(h_1(t))^2\right] \\ \frac{dh_2(t)}{dt} &= \frac{k_2}{A_2}\left[\frac{1}{2}\sqrt{h_1^0} - \frac{1}{8\left(\sqrt{h_1^0}\right)^3}\left(h_1^0\right)^2\right] \\ &+ \frac{k_2}{A_2}\left[\left(\frac{1}{2\sqrt{h_1^0}} + \frac{h_1^0}{4\left(\sqrt{h_1^0}\right)^3}\right)h_1(t)\right] \\ &+ \frac{k_2}{A_2}\left[-\frac{1}{8\left(\sqrt{h_1^0}\right)^3}(h_1(t))^2\right] \\ &+ \frac{k_3}{A_2}\left[-\frac{1}{2}\sqrt{h_2^0} + \frac{1}{8\left(\sqrt{h_2^0}\right)^3}\left(h_2^0\right)^2\right] \\ &+ \frac{k_3}{A_2}\left[-\left(\frac{1}{2\sqrt{h_2^0}} + \frac{h_2^0}{4\left(\sqrt{h_2^0}\right)^3}\right)h_2(t)\right] \\ &+ \frac{k_3}{A_2}\left[\frac{1}{8\left(\sqrt{h_2^0}\right)^3}(h_2(t))^2\right]\end{aligned} \quad (37)$$

This model can be simplified into

$$\begin{aligned}\frac{dh_1(t)}{dt} &= \frac{k_2}{A_1}\left[\frac{F_1(t)}{k_2} - \frac{3}{8}\sqrt{h_1^0} - \frac{3}{4\sqrt{h_1^0}}h_1(t)\right] \\ &+ \frac{k_2}{A_1}\left[\frac{1}{8\left(\sqrt{h_1^0}\right)^3}(h_1(t))^2\right] \\ \frac{dh_2(t)}{dt} &= \frac{k_2}{A_2}\left[\frac{3}{8}\sqrt{h_1^0} + \frac{3}{4\sqrt{h_1^0}}h_1(t)\right] \\ &\frac{k_2}{A_2}\left[-\frac{1}{8\left(\sqrt{h_1^0}\right)^3}(h_1(t))^2\right] \\ &+ \frac{k_3}{A_2}\left[-\frac{3}{8}\sqrt{h_2^0} - \frac{3}{4\sqrt{h_2^0}}h_2(t)\right] \\ &+ \frac{k_3}{A_2}\left[\frac{1}{8\left(\sqrt{h_2^0}\right)^3}(h_2(t))^2\right]\end{aligned} \quad (38)$$

Here, again set the valve coefficient k_2 as the only uncertain variable. At this point one can apply polynomial chaos expansion to the state variables h_1 and h_2, and the uncertain parameters k_2 to obtain the same PCT expansion as in Equation 24. However, the model in Equation 38 that was obtained using Taylor approximation up to second order, now includes nonlinear multiplication of three variables: $k_2h_1h_1$ and $k_3h_2h_2$, respectively. Now, by inserting Equation 24 into the model in Equation 37, or in other words presenting the model using Legendre polynomials $\phi_i(\xi)$, yields:

$$\begin{aligned}A_1\sum_{i=0}^{P}\frac{dh_{1,i}(t,\xi)}{dt}\phi_i(\xi) &= F_1(t) - \frac{3}{8}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\,\phi_i(\xi) \\ &-\frac{3}{4\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,\phi_i(\xi)\,\phi_j(\xi) \\ &+\frac{1}{8\left(\sqrt{h_1^0}\right)^3}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{l=0}^{P}k_{2,i}\,h_{1,j}(t)\,h_{1,l}(t)\,\phi_i(\xi)\,\phi_j(\xi)\,\phi_l(\xi) \\ A_2\sum_{i=0}^{P}\frac{dh_{2,i}(t,\xi)}{dt}\phi_i(\xi) &= \frac{3}{8}\sqrt{h_1^0}\sum_{i=0}^{P}k_{2,i}\,\phi_i(\xi) \\ &+\frac{3}{4\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P}k_{2,i}\,h_{1,j}(t)\,\phi_i(\xi)\,\phi_j(\xi) \\ &-\frac{1}{8\left(\sqrt{h_1^0}\right)^3}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{l=0}^{P}k_{2,i}\,h_{1,j}(t)\,h_{1,l}(t)\,\phi_i(\xi)\,\phi_j(\xi)\,\phi_l(\xi) \\ &-\frac{3}{8}k_3\sqrt{h_2^0} - \frac{3}{4\sqrt{h_2^0}}k_3\sum_{j=0}^{P}h_{2,j}(t)\,\phi_j(\xi) \\ &\frac{1}{8\left(\sqrt{h_2^0}\right)^3}k_3\sum_{i=0}^{P}\sum_{j=0}^{P}h_{2,i}(t)\,h_{2,j}(t)\,\phi_i(\xi)\phi_j(\xi)\end{aligned} \quad (39)$$

Now a 4^{th}-order tensor notation should be introduced in addition to that in Equation 6:

$$\langle\phi_i\phi_j\phi_l\phi_k\rangle \equiv \langle\phi_k^2\rangle D_{ijlk} \quad (40)$$

where an inner product is defined for Legendre polynomials with weighting factor 1 according to:

$$\langle \phi_i \phi_k \rangle = \int_{-1}^{1} \phi_i(\xi)\, \phi_k(\xi)\, 1\, d\xi$$
$$\langle \phi_i \phi_j \phi_k \rangle = \int_{-1}^{1} \phi_i(\xi)\phi_j(\xi)\, \phi_k(\xi)\, 1\, d\xi \tag{41}$$
$$\langle \phi_i \phi_j \phi_l \phi_k \rangle = \int_{-1}^{1} \phi_i(\xi)\phi_j(\xi)\phi_l(\xi)\, \phi_k(\xi)\, 1\, d\xi$$

In Equations 6 and 40, C_{ijk} and D_{ijlk} are 3^{rd} and 4^{th}-order tensors, respectively, that can be determined based on the knowledge of Legendre polynomial terms ϕ_i.

At this point it is possible to discretize the PCT expanded model in Equation 39 using Galerkin projection onto the polynomial chaos basis in Equation 24 to get:

$$\int_{-1}^{1} \phi_k^2\, d\xi\, \frac{dh_{1,k}(t)}{dt} = + \int_{-1}^{1} \frac{1}{A_1}\left[H_1 + H_2 + H_3\right] \tag{42}$$
$$H_1 = F_1(t) - \frac{3}{8}\sqrt{h_1^0}\sum_{i=0}^{P} k_{2,i}\, \phi_i(\xi)$$
$$H_2 = -\frac{3}{4\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P} k_{2,i}\, h_{1,j}(t)\, \phi_i(\xi)\, \phi_j(\xi)$$
$$H_3 = \frac{1}{8\left(\sqrt{h_1^0}\right)^3}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{l=0}^{P} k_{2,i}\, h_{1,j}(t)\, h_{1,l}(t)\, \phi_i(\xi)\, \phi_j(\xi)\, \phi_l(\xi)$$

$$\int_{-1}^{1} \phi_k^2\, d\xi\, \frac{dh_{2,k}(t)}{dt} = \int_{-1}^{1} \frac{1}{A_2}\left[H_4 + H_5 + H_6 + H_7 + H_8\right]\phi_k\, d\xi \tag{43}$$
$$H_4 = \frac{3}{8}\sqrt{h_1^0}\sum_{i=0}^{P} k_{2,i}\, \phi_i(\xi)$$
$$H_5 = \frac{3}{4\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P} k_{2,i}\, h_{1,j}(t)\, \phi_i(\xi)\, \phi_j(\xi)$$
$$H_6 = -\frac{1}{8\left(\sqrt{h_1^0}\right)^3}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{l=0}^{P} k_{2,i}\, h_{1,j}(t)\, h_{1,l}(t)\, \phi_i(\xi)\, \phi_j(\xi)\, \phi_l(\xi)$$
$$H_7 = -\frac{3}{8}k_3\sqrt{h_2^0} - \frac{3}{4\sqrt{h_2^0}}k_3\sum_{j=0}^{P} h_{2,j}(t)\, \phi_j(\xi)$$
$$H_8 = \frac{1}{8\left(\sqrt{h_2^0}\right)^3}k_3\sum_{i=0}^{P}\sum_{j=0}^{P} h_{2,i}(t)\, h_{2,j}(t)\, \phi_i(\xi)\phi_j(\xi)$$

Equations 42 and 43 represent the PCT expanded nonlinear two tanks model, where instead of two original model equations the expanded model now consists of $2(k+1)$ equations. These equations can be modified and rewritten in terms of a 3^{rd}-order tensor C_{ijk} , a 4^{th}-order tensor D_{ijlk}, and the Kronecker delta δ_{ik} using Equations 6, 40 and 5, respectively:

$$A_1\frac{dh_{1,k}(t)}{dt} = \frac{\int_{-1}^{1} F_1(t)\, \phi_k\, d\xi}{\int_{-1}^{1} \phi_k^2\, d\xi} - \frac{3}{8}\sqrt{h_1^0}\sum_{i=0}^{P} k_{2,i}\delta_{ik} - \frac{3}{4\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P} k_{2,i}\, h_{1,j}(t)\, C_{ijk} + \frac{1}{8\left(\sqrt{h_1^0}\right)^3}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{l=0}^{P} k_{2,i}\, h_{1,j}(t)\, h_{1,l}(t)\, D_{ijlk}$$
$$A_2\frac{dh_{2,k}(t)}{dt} = \frac{3}{8}\sqrt{h_1^0}\sum_{i=0}^{P} k_{2,i}\delta_{ik} + \frac{3}{4\sqrt{h_1^0}}\sum_{i=0}^{P}\sum_{j=0}^{P} k_{2,i}\, h_{1,j}(t)\, C_{ijk} - \frac{1}{8\left(\sqrt{h_1^0}\right)^3}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{l=0}^{P} k_{2,i}\, h_{1,j}(t)\, h_{1,l}(t)\, D_{ijlk} - \frac{\int_{-1}^{1} \frac{3}{8}k_3\sqrt{h_2^0}\phi_k\, d\xi}{\int_{-1}^{1} \phi_k^2\, d\xi} - \frac{3}{4\sqrt{h_2^0}}k_3\sum_{j=0}^{P} h_{2,j}(t)\, \delta_{jk} + \frac{1}{8\left(\sqrt{h_2^0}\right)^3}k_3\sum_{i=0}^{P}\sum_{j=0}^{P} h_{2,i}(t)\, h_{2,j}(t)\, C_{ijk} \tag{44}$$

A denominator $\int_{-1}^{1} \phi_k^2\, d\xi$ in Equations 42 and 44 that accounts for orthogonality of Galerkin projection was given for k up to order 2 in Table 1. The terms $\langle \phi_i \phi_j \phi_k \rangle$ or, alternatively, C_{ijk} were also calculated up to order 2 (total of $3^3 = 27$ terms) using Legendre polynomials and normalization factors from Table 1. These terms appear in Table 2.

The main difference between the PCT expanded model after applying a Galerkin projection that was developed in the previous section and given in Equation 27 and the one obtained for the second order Taylor approximation - Equation 44 - is the addition of the 4^{th}-order tensor D_{ijlk}. Its coefficients can be computed for different combinations of indexes i,j,l, and k using an inner product definition that was given in Equations 41 and 40:

$$\left\langle \phi_i \phi_j \phi_l \phi_k \right\rangle = \int_{-1}^{1} \phi_i(\xi)\phi_j(\xi)\phi_l(\xi)\, \phi_k(\xi)\, d\xi = \left\langle \phi_k^2 \right\rangle D_{ijlk} \tag{45}$$

whereas Legendre polynomials $\phi_i(\xi)$ were presented in Table 1 for $i = [0 \dots 2]$. It is, however, easier to use the fact that integration over an odd function under the limits $[-1, 1]$ always results in zero, which again eliminates all the tensor terms for which the sum $(i+j+l+k)$ is an odd number. Moreover, in cases when any one of the indexes i,$j$or l is zero, since $\phi_0(\xi) = 1$, using Equations 6 and 40 and choosing, for instance, $j = 0$:

$$\langle \phi_i \phi_0 \phi_l \phi_k \rangle \equiv \left\langle \phi_k^2 \right\rangle D_{ijlk} = \langle \phi_i \phi_l \phi_k \rangle \equiv \left\langle \phi_k^2 \right\rangle C_{ilk} \quad (46)$$

so that:

$$for\ j = 0: \qquad D_{ijlk} = C_{ilk} \qquad (47)$$

whereas the 3^{rd}-order tensor coefficients C_{ilk} in Equation 47 can be found in Table 2. In the first part of the resulting PCT expanded model (Equation 27) a term $\int_{-1}^{1} F_1(t)\,\phi_k d\xi$ contributes only when $k = 0$ due to the properties of Legendre polynomials. For the simplest case of zero-order PCT expansion, the resulting model consisting of two differential equations is:

$$\begin{aligned}
\frac{dh_{1,0}(t)}{dt} &= \frac{1}{A_1}\left[F_1(t) - \frac{3}{4\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t)\right] \\
&+ \frac{1}{A_1}\left[-\frac{3}{8}k_{2,0}\sqrt{h_1^0} + \frac{1}{8}k_{2,0}\frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3}\right] \\
\frac{dh_{2,0}(t)}{dt} &= \frac{1}{A_2}\left[\frac{3}{4\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t)\right] \\
&+ \frac{1}{A_2}\left[\frac{3}{8}k_{2,0}\sqrt{h_1^0} - \frac{3}{8}k_3\sqrt{h_2^0} + \frac{1}{8}k_3\frac{h_{2,0}(t)^2}{\left(\sqrt{h_2^0}\right)^3}\right] \\
&+ \frac{1}{A_2}\left[-\frac{3}{4\sqrt{h_2^0}}\,k_3 h_{2,0}(t) - \frac{1}{8}k_{2,0}\frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3}\right]
\end{aligned} \qquad (48)$$

For this zero-order expanded model, $C_{000} = 1$ and $D_{0000} = 1$ were used. Equation 48 can be analyzed as the PCT expanded system achieves steady-state. In this case, the derivatives of the expanded states will vanish which will result in two nonlinear equalities:

$$\begin{aligned}
&F_1^{ss} - \frac{3}{4\sqrt{h_1^0}}\,k_{2,0}\,h_1^{ss} - \frac{3}{8}k_{2,0}\sqrt{h_1^0} + \frac{1}{8}k_{2,0}\frac{\left(h_1^{ss}\right)^2}{\left(\sqrt{h_1^0}\right)^3} = 0 \\
&\frac{3}{4\sqrt{h_1^0}}\,k_{2,0}\,h_1^{ss} + \frac{3}{8}k_{2,0}\sqrt{h_1^0} - \frac{3}{8}k_3\sqrt{h_2^0} \\
&-\frac{3}{4\sqrt{h_2^0}}\,k_3 h_2^{ss} + \frac{1}{8}k_3\frac{\left(h_2^{ss}\right)^2}{\left(\sqrt{h_2^0}\right)^3} - \frac{1}{8}k_{2,0}\frac{\left(h_1^{ss}\right)^2}{\left(\sqrt{h_1^0}\right)^3} = 0
\end{aligned} \qquad (49)$$

Summing the two equalities in Equation 49 results in a more comfortable set

$$\begin{aligned}
&F_1^{ss} - \frac{3}{4\sqrt{h_1^0}}\,k_{2,0}\,h_1^{ss} - \frac{3}{8}k_{2,0}\sqrt{h_1^0} + \frac{1}{8}k_{2,0}\frac{\left(h_1^{ss}\right)^2}{\left(\sqrt{h_1^0}\right)^3} = 0 \\
&F_1^{ss} - \frac{3}{8}k_3\sqrt{h_2^0} - \frac{3}{4\sqrt{h_2^0}}\,k_3 h_2^{ss} + \frac{1}{8}k_3\frac{\left(h_2^{ss}\right)^2}{\left(\sqrt{h_2^0}\right)^3} = 0
\end{aligned} \qquad (50)$$

In both Equations 49 and 50, $h_1^{ss} = h_{1,0}(t \to \infty)$ and $h_2^{ss} = h_{2,0}(t \to \infty)$ are the steady-state values of the states, and $F_1^{ss} = F_1(t \to \infty)$ is the steady state value of the input. The equalities that appear in Equation 50 are second order polynomials that can be easily solved using traditional methods for a known steady state input value F_1^{ss}. The first-order PCT expanded model consists of four differential equations with up to $P = 1$ terms in each. Using the values of C_{ijk} and D_{ijlk} for $k = 0$ and $k = 1$, one can obtain from Equation 44:

$$\begin{aligned}
A_1\frac{dh_{1,0}(t)}{dt} &= F_1(t) - \frac{3}{4\sqrt{h_1^0}}\,k_{2,0}\,h_{1,0}(t) \\
&-\frac{3}{8}k_{2,0}\sqrt{h_1^0} + \frac{1}{8}k_{2,0}\frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} - \frac{1}{4}k_{2,1}\frac{h_{1,1}(t)}{\sqrt{h_1^0}} \\
&+\frac{1}{24}k_{2,0}\frac{h_{1,1}(t)^2}{\left(\sqrt{h_1^0}\right)^3} + \frac{1}{12}k_{2,1}\frac{h_{1,0}(t)\,h_{1,1}(t)}{\left(\sqrt{h_1^0}\right)^3} \\
A_1\frac{dh_{1,1}(t)}{dt} &= \frac{3}{40}k_{2,1}\frac{h_{1,1}(t)^2}{\left(\sqrt{h_1^0}\right)^3} - \frac{3}{4}k_{2,0}\frac{h_{1,1}(t)}{\sqrt{h_1^0}} \\
&+\frac{1}{4}k_{2,0}\frac{h_{1,0}(t)\,h_{1,1}(t)}{\left(\sqrt{h_1^0}\right)^3} - \frac{3}{4}k_{2,1}\frac{h_{1,0}(t)}{\sqrt{h_1^0}} \\
&+\frac{1}{8}k_{2,1}\frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} - \frac{3}{8}k_{2,1}\sqrt{h_1^0} \\
A_2\frac{dh_{2,0}(t)}{dt} &= -\frac{1}{24}k_{2,0}\frac{h_{1,1}(t)^2}{\left(\sqrt{h_1^0}\right)^3} + \frac{1}{4}k_{2,1}\frac{h_{1,1}(t)}{\sqrt{h_1^0}} \\
&+\frac{1}{24}k_3\frac{h_{2,1}(t)^2}{\left(\sqrt{h_2^0}\right)^3} - \frac{1}{12}k_{2,1}\frac{h_{1,0}(t)\,h_{1,1}(t)}{\left(\sqrt{h_1^0}\right)^3} \\
&+\frac{1}{8}k_3\frac{h_{2,0}(t)^2}{\left(\sqrt{h_2^0}\right)^3} - \frac{1}{8}k_{2,0}\frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} + \frac{3}{4}k_{2,0}\frac{h_{1,0}(t)}{\sqrt{h_1^0}} \\
&+\frac{3}{4}k_3\frac{h_{2,0}(t)}{\sqrt{h_2^0}} + \frac{3}{8}k_{2,0}\sqrt{h_1^0} - \frac{3}{8}k_3\sqrt{h_2^0} \\
A_2\frac{dh_{2,1}(t)}{dt} &= \frac{3}{8}k_{2,1}\sqrt{h_1^0} + \frac{3}{4}k_{2,0}\frac{h_{1,1}(t)}{\sqrt{h_1^0}} \\
&-\frac{3}{40}k_{2,1}\frac{h_{1,1}(t)^2}{\left(\sqrt{h_1^0}\right)^3} - \frac{1}{8}k_{2,1}\frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} \\
&-\frac{3}{4}k_3\frac{h_{2,1}(t)}{\sqrt{h_2^0}} + \frac{1}{4}k_3\frac{h_{2,0}(t)\,h_{2,1}(t)}{\left(\sqrt{h_2^0}\right)^3} \\
&+-\frac{1}{4}k_{2,0}\frac{h_{1,0}(t)\,h_{1,1}(t)}{\left(\sqrt{h_1^0}\right)^3} + \frac{3}{4}k_{2,1}\frac{h_{1,0}(t)}{\sqrt{h_1^0}}
\end{aligned} \qquad (51)$$

For this first-order expanded model, coefficients $D_{1001} = D_{0101} = D_{0011} = 1$ and $D_{1111} = \frac{3}{5}$ were used based on Equation 47 and the values of corresponding coefficients C_{ijk} from Table 2. Higher order PCT expanded models can be developed in a similar way. They include too many terms to be included in this paper.

Equation 51 represents a set of four nonlinear differential equations that can be presented in a general compact form:

$$\begin{aligned}
\frac{d\underline{x}_{PCT}(t)}{dt} &= \underline{f}_{PCT}\left(\underline{x}_{PCT}(t),\ \underline{u}_{PCT}(t)\right) \\
\underline{y}_{PCT}(t) &= \underline{h}_{PCT}\left(\underline{x}_{PCT}(t)\right)
\end{aligned} \qquad (52)$$

The uncertain model given in Equation 51 can be initially analyzed for the simplest case of no uncertainty in the variable k_2, i.e. $k_{2,1} = 0$. For this case, Equation 51 is reduced to:

$$
\begin{aligned}
A_1 \frac{dh_{1,0}(t)}{dt} &= F_1(t) - \frac{3}{4\sqrt{h_1^0}} k_{2,0}\, h_{1,0}(t) - \frac{3}{8} k_{2,0} \sqrt{h_1^0} \\
&+ \frac{1}{8} k_{2,0} \frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} + \frac{1}{24} k_{2,0} \frac{h_{1,1}(t)^2}{\left(\sqrt{h_1^0}\right)^3} \\
A_1 \frac{dh_{1,1}(t)}{dt} &= -\frac{3}{4} k_{2,0} \frac{h_{1,1}(t)}{\sqrt{h_1^0}} + \frac{1}{4} k_{2,0} \frac{h_{1,0}(t)\, h_{1,1}(t)}{\left(\sqrt{h_1^0}\right)^3} \\
A_2 \frac{dh_{2,0}(t)}{dt} &= -\frac{1}{24} k_{2,0} \frac{h_{1,1}(t)^2}{\left(\sqrt{h_1^0}\right)^3} + \frac{1}{24} k_3 \frac{h_{2,1}(t)^2}{\left(\sqrt{h_2^0}\right)^3} \\
&+ \frac{3}{4} k_{2,0} \frac{h_{1,0}(t)}{\sqrt{h_1^0}} + \frac{1}{8} k_3 \frac{h_{2,0}(t)^2}{\left(\sqrt{h_2^0}\right)^3} + \frac{3}{8} k_{2,0} \sqrt{h_1^0} \\
&- \frac{1}{8} k_{2,0} \frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} - \frac{3}{4} k_3 \frac{h_{2,0}(t)}{\sqrt{h_2^0}} - \frac{3}{8} k_3 \sqrt{h_2^0} \\
A_2 \frac{dh_{2,1}(t)}{dt} &= \frac{3}{4} k_{2,0} \frac{h_{1,1}(t)}{\sqrt{h_1^0}} - \frac{1}{4} k_{2,0} \frac{h_{1,0}(t)\, h_{1,1}(t)}{\left(\sqrt{h_1^0}\right)^3} \\
&- \frac{3}{4} k_3 \frac{h_{2,1}(t)}{\sqrt{h_2^0}} + \frac{1}{4} k_3 \frac{h_{2,0}(t)\, h_{2,1}(t)}{\left(\sqrt{h_2^0}\right)^3}
\end{aligned} \tag{53}
$$

Moreover, if it is assumed that all the parameters are certain it would make sense to further simplify the model by eliminating the states distribution terms by setting $h_{1,1}(t) = 0$ and $h_{2,1}(t) = 0$, so that Equation 53 becomes

$$
\begin{aligned}
\frac{dh_{1,0}(t)}{dt} &= \frac{1}{A_1} \left[F_1(t) - \frac{3}{4\sqrt{h_1^0}} k_{2,0}\, h_{1,0}(t) \right] \\
&+ \frac{1}{A_1} \left[-\frac{3}{8} k_{2,0} \sqrt{h_1^0} + \frac{1}{8} k_{2,0} \frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} \right] \\
\frac{dh_{2,0}(t)}{dt} &= \frac{1}{A_2} \left[\frac{3}{4} k_{2,0} \frac{h_{1,0}(t)}{\sqrt{h_1^0}} + \frac{1}{8} k_3 \frac{h_{2,0}(t)^2}{\left(\sqrt{h_2^0}\right)^3} + \frac{3}{8} k_{2,0} \sqrt{h_1^0} \right] \\
&+ \frac{1}{A_2} \left[-\frac{1}{8} k_{2,0} \frac{h_{1,0}(t)^2}{\left(\sqrt{h_1^0}\right)^3} - \frac{3}{4} k_3 \frac{h_{2,0}(t)}{\sqrt{h_2^0}} - \frac{3}{8} k_3 \sqrt{h_2^0} \right]
\end{aligned} \tag{54}
$$

which in turn is identical to the first-order PCT model given in Equation 48. Equation 51 can be analyzed for the steady-state constraints. At steady state all the derivatives in Equation must vanish which results in a set of four equality constraints in which the variables $h_{1,0}$, $h_{1,1}$, $h_{2,0}$, $h_{2,1}$ are the values of the expanded states at steady state. Given all the constant parameters and the steady-state input value F_1 the task of evaluating steady-state constraints for this first-order expanded PCT model reduces to solving a set of four second-order algebraic equations with four unknowns $h_{1,0}$, $h_{1,1}$, $h_{2,0}$, $h_{2,1}$.

5. OPEN-LOOP RESULTS

The original system model was introduced in Equation 20. To solve that model one needs to make sure that the system is open-loop stable. The eigenvalues of the system given in Equation 20 are $\lambda_1 = -\frac{k_2}{A_1}$ and $\lambda_2 = -\frac{k_3}{A_2}$. To ensure open-loop stability, i.e. both eigenvalues less than -1, can select the following values for the parameters used in the model: $A_1 = 1$ [units of area], $A_2 = 2$ [units of $area$], $k_2 = 3$ [units of $\frac{area}{time}$], and $k_3 = 4$ [units of $\frac{area}{time}$]. Substituting these values into the original model in Equation 20, one gets:

$$
\begin{aligned}
\frac{dh_1(t)}{dt} &= F_1(t) - 3\sqrt{h_1(t)} \\
\frac{dh_2(t)}{dt} &= 1.5\sqrt{h_1(t)} - 2\sqrt{h_2(t)}
\end{aligned} \tag{55}
$$

Equation 55 is an open-loop stable system of two nonlinear differential equations, for which one can assume zero initial states: $h_1(0) = h_2(0) = 0$. The steady-state values, $h_{1,ss}$ and $h_{2,ss}$ can be easily obtained from Equation 55 using $\frac{dh_i(t)}{dt} = 0$. For a steady state input value $F_1^{ss} = 12$ [units of $\frac{volume}{time}$] the result is: $h_1^{ss} = 16$ [units of height], and $h_2^{ss} = 9$ [units of height]. To obtain the solution of the model in Equation 55, one can linearize the model with respect to steady state values as shown below using first-order Taylor approximation.

5.1 *Zero-Order PCT Expansion*

The simplest zero-order PCT expanded model was derived earlier and is given in Equation 28. For the given problem with only one uncertain parameter k_2 it is expanded using Legendre polynomials up to order 0 according to:

$$
k_2 = k_{2,0}\, \phi_0(\xi) = k_{2,0} \tag{56}
$$

in which $\phi_0(\xi) = 1$ was introduced earlier in Table 1. A uniformly distributed (dimensionless) random variable ξ does not appear in the final expanded model given in Equation 29, and is only used for PCT expansion as was shown in the previous section. With analogy to the original problem where a parameter k_2 was assumed to have a steady state value of 3, one can choose a mean value $k_{2,0} = 3$ [units of $\frac{area}{time}$]. Substituting this mean value together with all the other parameters into Equation 28, yields the following zero-order PCT expanded model:

$$
\begin{aligned}
\frac{dh_{1,0}(t)}{dt} &= \left[F_1(t) - \frac{3}{2\sqrt{h_1^0}} h_{1,0}(t) - \frac{3}{2}\sqrt{h_1^0} \right] \\
\frac{dh_{2,0}(t)}{dt} &= \\
\frac{1}{2} &\left[\frac{3}{2\sqrt{h_1^0}} h_{1,0}(t) + \frac{3}{2}\sqrt{h_1^0} - 2\sqrt{h_2^0} - \frac{2}{\sqrt{h_2^0}} h_{2,0}(t) \right]
\end{aligned} \tag{57}
$$

This zero-order PCT expanded model that assumes no distribution in the uncertain parameter k_2 depends on the starting points h_1^0 and h_2^0 that were used for Taylor approximation. Since this work addresses deviations from steady conditions, it is assumed that Taylor approximation applied around the steady state points h_1^{ss} and h_2^{ss} so that for a steady state input value $F_1^{ss} = 12$ [units of $\frac{volume}{time}$]:

$$h_1^0 = h_1^{ss} = 16$$
$$h_2^0 = h_2^{ss} = 9 \tag{58}$$

[units of height] and the Taylor approximation up to order 1 in Equation 22 reduces to

$$\sqrt{h_1} = 2 + \frac{1}{8}\, h_1$$
$$\sqrt{h_2} = \frac{3}{2} + \frac{1}{6}\, h_2 \tag{59}$$

Now, if one inserts Equation 59 into the original nonlinear model in Equation 55, the latter becomes:

$$\frac{dh_1(t)}{dt} = \frac{1}{2}\, F_1\,(t) - \frac{3}{8}\, h_1(t)$$
$$\frac{dh_2(t)}{dt} = \frac{3}{16}\, h_1(t) - \frac{1}{3}\, h_2(t) \tag{60}$$

Substituting the values from Equation 58 into Equation 57, yields:

$$\frac{dh_{1,0}(t)}{dt} = \frac{1}{2}\, F_1\,(t) - \frac{3}{8}\, h_{1,0}(t)$$
$$\frac{dh_{2,0}(t)}{dt} = \frac{3}{16}\, h_{1,0}(t) - \frac{1}{3}\, h_{2,0}(t) \tag{61}$$

As expected, the two resulting models in Equations 60 and 62 are identical, for in the simplest zero-order PCT expanded case $h_i = h_{i,0}$. To obtain the solution of the linear model in Equation 60, can implement Laplace transform of each differential equation to get:

$$s\, H_1(s) - h_1(0) = \frac{1}{2} F_1\,(s) - \frac{3}{8}\, H_1\,(s)$$
$$s\, H_2(s) - h_2(0) = \frac{3}{16}\, H_1\,(s) - \frac{1}{3}\, H_2\,(s) \tag{62}$$

Eliminating the initial conditions $h_1\,(0) = h_2\,(0) = 0$ Equation 62 is simplified to

$$H_1\,(s) = \frac{1}{2} F_1\,(s)\, \frac{1}{\left(s + \frac{3}{8}\right)}$$
$$H_2\,(s) = \frac{3}{32} F_1\,(s)\, \frac{1}{\left(s + \frac{3}{8}\right)\left(s + \frac{1}{3}\right)} \tag{63}$$

where $H_i(s)$ and $F_1(s)$ are Laplace transforms in s-domain of $h_i(t)$ and $F_1(t)$, respectively. Given a constant steady state input value $F_1^{ss} = 12$ [units of $\frac{volume}{time}$] and applying the inverse Laplace transform on Equation 63, one obtains the solution to the original (or zero-order PCT) linearized model:

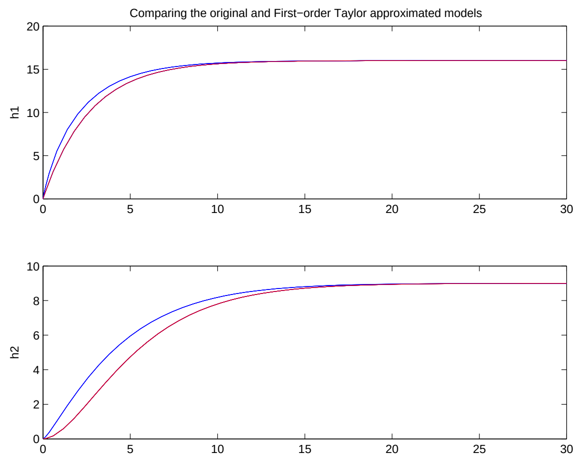

Fig. 2. Comparison between the original model and the First-order Taylor approximated model of two tanks.

$$h_1(t) = 16 - 16e^{-\frac{3}{8}t}$$
$$h_2(t) = 9 - 81e^{-\frac{1}{3}t} + 72e^{-\frac{3}{8}t} \tag{64}$$

Figure 2 shows the comparison between the original nonlinear two tank model (blue) and the First-order Taylor approximated model (red).

5.2 *First-Order PCT Expansion*

The first order PCT expanded model was also derived earlier and given in Equation 29. For the given problem with only one uncertain parameter k_2 it is expanded using Legendre polynomial up to order 1 according to:

$$k_2 = k_{2,0}\phi_0(\xi) + k_{2,1}\phi_1(\xi) = k_{2,0} + k_{2,1}\xi \tag{65}$$

in which $\phi_0(\xi) = 1$ and $\phi_1(\xi) = \xi$ were introduced earlier in Table 1. Similar to the previous case of zero-order PCT expansion, where a mean value of $k_{2,0} = 3$ [units of $\frac{area}{time}$] was used, now a first-order distribution value of $k_{2,1} = 0.3$ [units of $\frac{area}{time}$] is being used to introduce disturbance around the mean value in the uncertain parameter k_2. The starting points for Taylor approximation are also chosen as in Equation 58. Substituting all the values into Equation 29, yields the following first-order PCT expanded model:

$$\frac{dh_{1,0}(t)}{dt} = \frac{1}{2}\left[2F_1\,(t) - \frac{3}{4}\, h_{1,0}(t) - \frac{1}{40}\, h_{1,1}(t) - 12\right]$$
$$\frac{dh_{1,1}(t)}{dt} = \frac{1}{2}\left[-\frac{3}{4}\, h_{1,1}(t) - \frac{3}{40}\, h_{1,0}(t) - 1.2\right]$$
$$\frac{dh_{2,0}(t)}{dt} = \frac{1}{4}\left[\frac{3}{4}\, h_{1,0}(t) + \frac{1}{40}\, h_{1,1}(t) - \frac{4}{3}\, h_{2,0}(t)\right]$$
$$\frac{dh_{2,1}(t)}{dt} = \frac{1}{4}\left[\frac{3}{4}\, h_{1,1}(t) + \frac{3}{40}\, h_{1,0}(t) - \frac{4}{3}\, h_{2,1}(t) + 1.2\right] \tag{66}$$

Equation 66 now consists of four differential equations (with assumed zero initial conditions) that

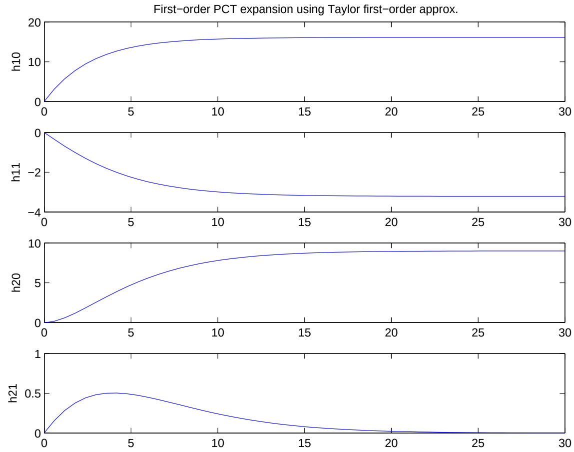

Fig. 3. First-order PCT expanded model showing the states mean values and distributions for $F_1 = 12$.

can be solved using Matlab MathWorks (2000) as a function of the process input F_1. Equation 66 can be rewritten in a state-space format presented in Equation 67 as:

$$\begin{aligned} \frac{d\underline{x}_{PCT}(t)}{dt} &= A_{PCT}(t)\,\underline{x}_{PCT}(t) \\ &+B_{PCT}(t)\,\underline{u}(t) + \Gamma_{PCT} \\ \underline{y}_{PCT}(t) &= C_{PCT}(t)\,\underline{x}_{PCT}(t) \\ &+D_{PCT}(t)\,\underline{u}(t) \end{aligned} \tag{67}$$

where $\underline{x}_{PCT}(t) = [h_{1,0}(t) \quad h_{1,1}(t) \quad h_{2,0}(t) \quad h_{2,1}(t)]^T$ is a vector of expanded states, and $\underline{y}_{PCT}(t) = \underline{x}_{PCT}(t)$ is the output vector that includes all the expanded states. Matrices A_{PCT}, B_{PCT}, C_{PCT}, C_{PCT} and Γ_{PCT} are identified as follows:

$$A_{PCT} = \begin{pmatrix} -\frac{3}{8} & -\frac{1}{80} & 0 & 0 \\ -\frac{3}{80} & -\frac{3}{8} & 0 & 0 \\ \frac{3}{16} & \frac{1}{160} & -\frac{1}{3} & 0 \\ \frac{3}{160} & \frac{3}{16} & 0 & -\frac{1}{3} \end{pmatrix} \tag{68}$$

$$B_{PCT} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Gamma_{PCT} = \begin{pmatrix} -6 \\ -0.6 \\ 0 \\ 0.3 \end{pmatrix} \tag{69}$$

$$C_{PCT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_{PCT} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{70}$$

Figure 3 shows the states mean values $h_{1,0}$ and $h_{2,0}$ and their respective distributions $h_{1,1}$ and $h_{2,1}$ as they achieve the steady state. At this point, it might be useful to apply the same disturbance to the original problem (with no application of Polynomial Chaos Theory). This can be done in Matlab.

Figure 4 shows the states h_1 and h_2 for two cases: the original problem of a constant value of a parameter $k_2 = 3$ [units of $\frac{area}{time}$] (shown in blue) and the case when this parameter is subject to a random normal distribution of about 10% around its mean value (shown in red).

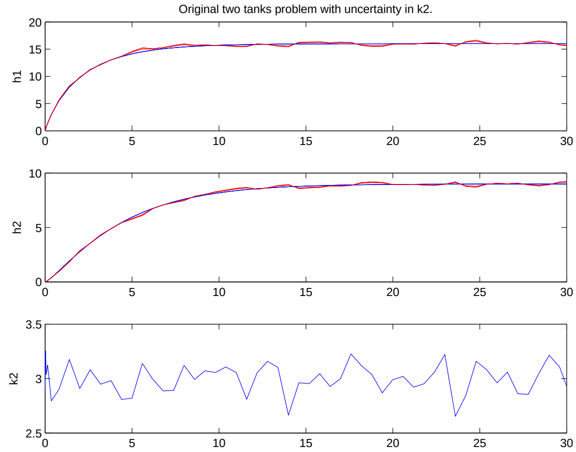

Fig. 4. The effect of uncertainty in the input parameter k_2 for the original problem.

6. CLOSED-LOOP RESULTS

This section examines the closed-loop results for control of the two-tank model. Initially, NMPC control formulation was applied on the original two-tank model with uncertain valve coefficient k_2. The nonsquare $2x1$ control system is examined using NMPC controller formulation and the proposed soft constraint methodology. The results appear in Figures 5 and 6. The fact that the original model used in the case-study for this work consists of two outputs and only one input makes it impossible for the controller to track all possible setpoints. Moreover, only certain ratio of the steady-state values can be achieved in this case. Therefore, the authors were more interested in analyzing the ability to apply soft constraints on the process outputs and thus enable safe operation within the bounds of interest even when a small disturbance is applied. By tuning the controller parameters such as penalties in the cost function it is possible to push the process into the desired region or make it track a certain setpoint if absolutely needed.

6.1 *NMPC with soft constraints on the process outputs applied on the original two-tank model*

Figure 5 shows closed-loop control of the original model with soft constraints on the outputs. This control operation used zero setpoint tracking penalties and equal soft constraints penalties for the two outputs. As can be observed from Figure 5 the controller is able to handle soft constraints on the outputs until a small disturbance hits the process at time $t = 18$. The disturbance causes

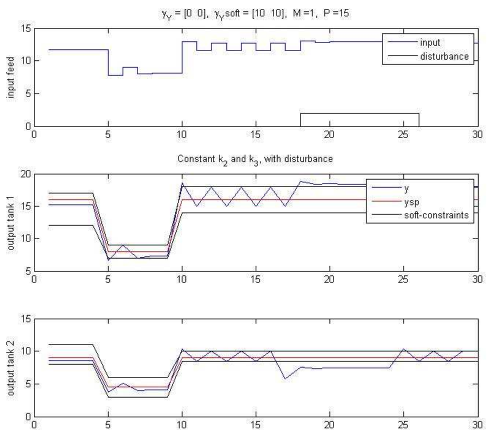

Fig. 5. NMPC with soft constraints on the process outputs with zero steady-state regulation penalty in the objective function. Observed no setpoint tracking and no soft-constraints violation until a small disturbance was applied and after its effect was removed.

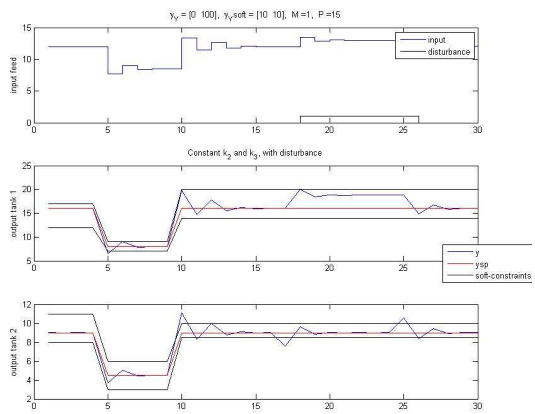

Fig. 6. NMPC with soft constraints on the process outputs with a large steady-state regulation penalty on the second output in the objective function. Shown setpoint tracking and no soft constraints violation even when a small disturbance was applied.

the process to deviate from the previous bounds up until its removal. A smaller disturbance is then tested with the same setpoints and slightly enhanced soft constraints bounds on the outputs while the tuning is set to track the setpoint of the second output. The results appear in Figure 6. It follows from the plot that no deviation from the safety bounds is observed despite the disturbance being applied. It is worth noting that the uncertain parameter k_2 was simulated to randomly change within a small region around its steady-state value.

6.2 *NMPC with soft constraints on the process outputs applied on the PCT expanded model*

As was described earlier in the paper PCT analysis when used for control purposes enables direct control of various components of the expanded model. In particular, a possible control tuning strategy when trying to decrease the effect of parametric uncertainty on the system's performance might include an increased penalty on those components that represent uncertainty. The first-order PCT expanded two-tank model with one uncertain parameter k_2, used with second-order Taylor approximation, was developed in Section 4 and resulted in a dynamic nonlinear model with four states given in Equation 51. Two of the states represent the mean components of the original outputs while the other two indicate the deviation from the mean values. A $4x1$ control system is formulated for this expanded model with the feed into the first tank being the only manipulated input.

Different control strategies can be implemented at this point by changing the tuning parameters. Obviously, for a given nonzero uncertainty in parameter k_2, based on the PCT analysis developed in the previous chapters it is practically impossible to completely minimize distribution components without shutting down the flow into the system. In general control problems of this type a hard input constraint can be used to eliminate this natural optimization solution. In order to fulfill the research goals of this work, the authors tried to implement a control strategy that enables to apply soft constraints on all the expanded outputs thus decreasing the risk of process deviation from the safety or economically reasoned bounds.

Figure 7 represents the case for which the penalties in the cost function are tuned to maintain the mean components within certain bounds, while the components associated with uncertainty are unbound. Figure 8, alternatively, describes the case when only the distribution components are bound to certain limits. A comparison between the plots leads to a conclusion that it is possible to slightly decrease the uncertainty in the outputs using the proposed control methodology without changing the parametric uncertainty of the system.

7. CONCLUSIONS

Polynomial Chaos Theory analysis can be effectively applied on nonlinear systems with uncertain parameters. The main advantage of using PCT in a time domain lies in the ability to analytically obtain the expanded solution in a single computational run. The expanded system

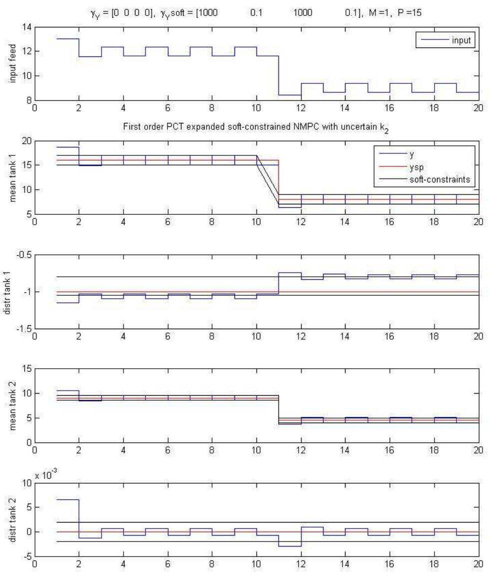

Fig. 7. NMPC with soft constraints on the mean component of process outputs using a first-order PCT expanded model with one uncertain parameter k_2.

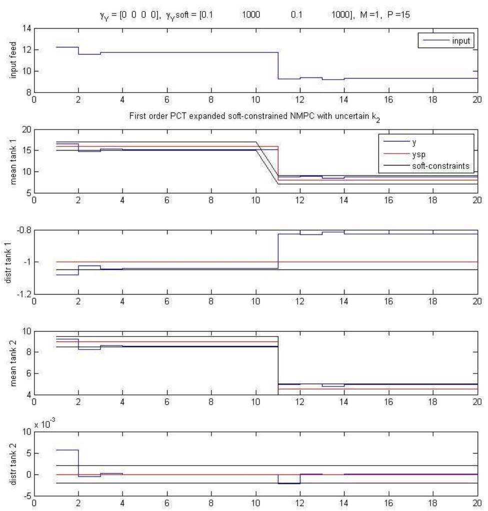

Fig. 8. NMPC with soft constraints on the distribution components of process outputs using a first-order PCT expanded model with one uncertain parameter k_2.

increases the number of the original states in the system depending on the order of PCT expansion and the number of uncertain parameters, so that the resulting dynamic PCT expanded model provides outputs that represent mean and distribution components of the original states. Provided the original distribution of the random variables these components can be reorganized to represent the original states. The multivariable expanded solution can be used along with the proposed Nonlinear Model Predictive Control formulation to control the individual outputs of the PCT expanded model. The use of NMPC controller can decrease the sensitivity of the model to changes in the uncertain parameters by applying large penalties in the cost function on those components that represent uncertainty. Application of soft constraints on the PCT expanded process outputs enables safe operation within certain bounds.

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