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BOUNDARY CONTROL OF AN INFINITE ORDER TIME DELAY PARABOLIC SYSTEM WITH NON-DIFFERENTIABLE PERFORMANCE FUNCTIONAL

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Abstract: Various optimization problems associated with the optimal control of distributed-parameter systems with time delays appearing in the boundary conditions have been studied recently in Kowalewski (1988), Kowalewski (1990), Kowalewski (1998), Kowalewski and Duda (1992), Wang (1975) and Wong (1987). In this paper, we consider an optimal boundary control problem for an infinite order parabolic system with time delay given in the integral form. Sufficient conditions for the existence of a unique solution of the infinite order parabolic delay equation with the Neumann boundary condition involving a time delay in the integral form are proved. The performance functional constitutes the sum of a differentiable and non-differentiable function. The time horizon T is fixed. Finally, we impose some constraints on the control. Making use of the Lions scheme (Lions (1971)), necessary and sufficient conditions of optimality for the Neumann problem are derived.

Keywords: Boundary control, infinite order parabolic system, time delay, non-differentiable performance functional.

1. PRELIMINARIES

Let Ω be a bounded open set of R^n with smooth boundary Γ.

We define the infinite order Sobolev space $H^\infty\{a_\alpha, 2\}(\Omega)$ of functions $\Phi(x)$ defined on Ω Dubinskij (1975) and Dubinskij (1976) as follows

$$H^\infty\{a_\alpha, 2\}(\Omega) = \\ = \left\{ \Phi(x) \in C^\infty(\Omega) : \sum_{|\alpha|=0}^{\infty} a_\alpha \parallel \mathcal{D}^\alpha \Phi \parallel_2^2 < \infty \right\} \quad (1)$$

where: $C^\infty(\Omega)$ is a space of infinite differentiable functions, $a_\alpha \geq 0$ is a numerical sequence and $\| \cdot \|_2$ is a norm in the space $L^2(\Omega)$, and

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \dots (\partial x_n)^{\alpha_n}} \ , \quad (2)$$

where: $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

The space $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ (Dubinskij (1975) and Dubinskij (1976)) is defined as the formal conjugate space to the space $H^\infty\{a_\alpha, 2\}(\Omega)$, namely:

$$H^{-\infty}\{a_\alpha, 2\}(\Omega) = \\ = \left\{ \Psi(x) : \Psi(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathcal{D}^\alpha \Psi_\alpha(x) \right\} \quad (3)$$

where: $\Psi_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \parallel \Psi_\alpha \parallel_2^2 < \infty$.

The duality pairing of the spaces $H^\infty\{a_\alpha, 2\}(\Omega)$ and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ is postulated by the formula

$$\langle \Phi, \Psi \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega \Psi_\alpha(x) \mathcal{D}^\alpha \Phi(x)\, dx, \quad (4)$$

where: $\Phi \in H^\infty\{a_\alpha, 2\}(\Omega)$, $\Psi \in H^{-\infty}\{a_\alpha, 2\}(\Omega)$.

From above, $H^\infty\{a_\alpha, 2\}(\Omega)$ is everywhere dense in $L^2(\Omega)$ with topological inclusions and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ denotes the topological dual space with respect to $L^2(\Omega)$ so we have the following chain:

$$H^\infty\{a_\alpha, 2\}(\Omega) \subseteq L^2(\Omega) \subseteq H^{-\infty}\{a_\alpha, 2\}(\Omega). \quad (5)$$

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Consider now the distributed-parameter system described by the infinite order parabolic delay equation

$$\frac{\partial y}{\partial t} + Ay = u \quad x \in \Omega,\ t \in (0, T) \quad (6)$$

$$y(x, 0) = y_0(x) \quad x \in \Omega \quad (7)$$

$$\frac{\partial y}{\partial \eta_A}(x, t) = \int_a^b c(x, t) y(x, t - h) dh + v \quad x \in \Gamma,\ t \in (0, T) \quad (8)$$

$$y(x, t') = \Psi_o(x, t') \qquad x \in \Gamma, t' \in [-b, 0) \quad (9)$$

where: Ω has the same properties as in the Section 2.

$$y \equiv y(x, t; v), \quad u \equiv u(x, t), \quad v \equiv v(x, t)$$

$$Q = \Omega \times (0, T), \quad \bar{Q} = \bar{\Omega} \times [0, T],$$

$$\Sigma = \Gamma \times (0, T), \quad \Sigma_0 = \Gamma \times [-b, 0)$$

c is a given real C^∞ function defined on Σ,

h is a time delay such that $h \in (a, b)$,

Ψ_0 is an initial function defined on Σ_0.

The operator $\frac{\partial}{\partial t} + A$ in the state equation (6) is an infinite order parabolic operator and A (Dubinskij (1986)) is given by

$$Ay = \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathcal{D}^{2\alpha} + 1 \right) y \quad (10)$$

and

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathcal{D}^{2\alpha} \quad (11)$$

is an infinite order elliptic partial differential operator.

The operator A is a mapping of $H^\infty\{a_\alpha, 2\}(\Omega)$ onto $H^{-\infty}\{a_\alpha, 2\}(\Omega)$. For this operator the bilinear form $\Pi(t; y, \varphi) = (Ay, \varphi)_{L^2(\Omega)}$ is coercive on $H^\infty\{a_\alpha, 2\}(\Omega)$ i.e. there exists $\lambda > 0, \lambda \in \mathbb{R}$ such that $\Pi(t; y, \varphi) \geq \lambda \|y\|^2_{H^\infty\{a_\alpha,2\}(\Omega)}$ and $\forall\, y, \varphi \in H^\infty\{a_\alpha, 2\}(\Omega)$ the function $t \to \Pi(t; y, \varphi)$ is measurable on $[0, T]$.

The equations (6) - (9) constitute a Neumann problem. The left-hand side of (8) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = \sum_{|w|=0}^{\infty} (\mathcal{D}^w y(v)) \cos(n, x_i) = q(x, t) \quad x \in \Gamma,\ t \in (0, T) \quad (12)$$

where: $\frac{\partial}{\partial \eta_A}$ is a normal derivative at Γ, directed towards the exterior of $\Omega, \cos(n, x_i)$ is an i-th direction cosine of n, n - being the normal at Γ exterior to Ω and

$$q(x, t) = \int_a^b c(x, t) y(x, t - h) dh + v(x, t) \quad (13)$$

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (6) - (9) for the case where $v \in L^2(\Sigma)$.

For this purpose, we introduce the Sobolev space $H^{\infty,1}(Q)$ (Lions and Magenes (1972), Vol. 2, p. 6) defined by

$$H^{\infty,1}(Q) = \quad (14)$$
$$= H^0(0, T; H^\infty\{a_\alpha, 2\}(\Omega)) \cap H^1(0, T; H^0(\Omega))$$

which is a Hilbert space normed by

$$\left(\int_0^T \left\| y(t) \right\|^2_{H^\infty\{a_\alpha,2\}(\Omega)} dt + \left\| y \right\|^2_{H^1(0,T;H^0(\Omega))} \right)^{\frac{1}{2}} \quad (15)$$

where: the space $H^1(0, T; H^0(\Omega))$ is defined in Chapter 1 (Lions and Magenes (1972), Vol.1).

The existence of a unique solution for the mixed initial-boundary value problem (6) - (9) on the cylinder Q can be proved using a constructive method, i.e., first, solving (6) - (9) on the subcylinder Q_1 and in turn on Q_2, etc. until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

For simplicity, we introduce the following notations:

$T = Ka$ where K - a positive integer, and

$$E_j \triangleq ((j-1)a, ja), \qquad Q_j = \Omega \times E_j,$$

$$\Sigma_j = \Gamma \times E_j \text{ for } j = 1, \ldots, K$$

Using the Theorem 15.2 (Lions and Magenes (1972), Vol.2, p. 81) we can prove the following lemma.

Lemma 1. Let

$$u \in (H^{\infty,1}(Q))', \quad v \in L^2(\Sigma) \tag{16}$$

$$y_{j-1}(\cdot,(j-1)a) \in H^{\infty}\{a_\alpha, 2\}(\Omega) \tag{17}$$

$$q_j \in L^2(\Sigma_j) \tag{18}$$

where

$$q_j(x,t) = \int_a^b c(x,t) y_{j-1}(x, t-h) dh + v(x,t)$$

Then, there exists a unique solution $y_j \in H^{\infty,1}(Q_j)$ for the mixed initial-boundary value problem (6), (8), (17).

Proof: We observe that for $j = 1$, $y_{j-1}|_{\Sigma_0}(x, t-h) = \Psi_0(x, t-h)$. Then the assumptions (17) and (18) are fulfilled if we assume that $y_0 \in H^{\infty}\{a_\alpha, 2\}(\Omega)$ and $\Psi_0 \in L^2(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in H^{\infty,1}(Q_1)$. Next for $j = 2$ we have to verify that $y_1(\cdot, a) \in H^{\infty}\{a_\alpha, 2\}(\Omega)$ and $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Then using the Theorem 3.1 (Lions and Magenes (1972), Vol.1, p.19) we can prove that $y_1 \in H^{\infty,1}(Q_1)$ implies that the mapping $t \to y_1(\cdot, t)$ is continuous from $[0, a] \to H^{\infty}\{a_\alpha, 2\}(\Omega)$, hence $y_1(\cdot, a) \in H^{\infty}\{a_\alpha, 2\}(\Omega)$. Again from the Trace Theorem (Theorem 2.1 in Lions and Magenes (1972), Vol.2, p. 9) $y_1 \in H^{\infty,1}(Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping of $H^{\infty,1}(Q_1) \to H^{\infty,1}(\Sigma_1)$. Thus, $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Then, there exists a unique solution $y_2 \in H^{\infty,1}(Q_2)$. We shall now summarize the foregoing result for any Q_j, $j = 3, \ldots, K$.

Theorem 1. Let y_0, Ψ_0, v and u be given with $y_0 \in H^{\infty}\{a_\alpha, 2\}(\Omega), \Psi_0 \in L^2(\Sigma_0), v \in L^2(\Sigma)$ and $u \in (H^{\infty,1}(Q))'$. Then, there exists a unique solution $y \in H^{\infty,1}(Q)$ for the mixed initial-boundary value problem (6) - (9). Moreover, $y(\cdot, ja) \in H^{\infty}\{a_\alpha, 2\}(\Omega)$ for $j = 1, \ldots, K$.

3. PROBLEM FORMULATION. OPTIMIZATION THEOREMS

We shall now formulate the optimal boundary control problem for the Neumann problem. Let us denote by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \int_Q | y(x,t;v) - z_d |^2 \, dxdt + \tag{19}$$
$$+ \lambda_2 \int_\Sigma (Nv) v d\Gamma dt + 2\lambda_3 \int_\Sigma |v| d\Gamma dt$$

where: $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 > 0$; z_d is a given element in $L^2(Q)$; N is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$.

Finally, we assume the following constraint on controls $v \in U_{ad}$, where

$$U_{ad} \text{ is a closed, convex subset of } U \tag{20}$$

Let $y(x,t;v)$ denote the solution of the mixed initial-boundary value problem (6) - (9) at (x,t) corresponding to a given control $v \in U_{ad}$. We note from the Theorem 1 that for any $v \in U_{ad}$ the performance functional (19) is well-defined since $y(v) \in H^{\infty,1}(Q) \subset L^2(Q)$. The solving of the formulated optimal control problem is equivalent to seeking a $v_0 \in U_{ad}$ such that $I(v_0) \leq I(v) \forall v \in U_{ad}$.

Then from the Theorem 1.6 (Lions (1971), p. 12) it follows that for $\lambda_2 > 0$ and $\lambda_3 > 0$ a unique optimal control v_0 exists; moreover, v_0 is characterized by the following condition

$$I_1'(v_0)(v - v_0) + I_2(v) - I_2(v_0) \geq 0 \;\forall v \in U_{ad} \tag{21}$$

where: $I_1(v)$ is a differentiable function, $I_2(v)$ is not necessarily differentiable function.

Using the form of the performance functional (19) we can express (21) in the following form

$$\lambda_1 \int_Q (y(v_0) - z_d)(y(v) - y(v_0)) dxdt +$$
$$+ \lambda_2 \int_\Sigma N v_0 (v - v_0) d\Gamma dt + \lambda_3 \int_\Sigma |v| d\Gamma dt +$$
$$-\lambda_3 \int_\Sigma |v_0| d\Gamma dt \geq 0 \tag{22}$$

To simplify (22), we introduce the adjoint equation and for every $v \in U_{ad}$, we define the adjoint variable $p = p(v) = p(x,t;v)$ as the solution of the equation

$$-\frac{\partial p(v)}{\partial t} + A^* p(v) = \lambda_1 (y(v) - z_d) \quad x \in \Omega,\ t \in (0,T) \tag{23}$$

$$p(x,T;v) = 0 \qquad x \in \Omega \tag{24}$$

$$\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = \int_a^b c(x,t+h) p(x,t+h;v) dh \quad x \in \Gamma,\ t \in (0, T-b) \tag{25}$$

$$\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = \int_a^{T-t} c(x,t+h) p(x,t+h;v) dh \quad x \in \Gamma,\ t \in (T-b, T-a) \tag{26}$$

$$\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = 0 \qquad x \in \Gamma,\ t \in (T-a, T) \tag{27}$$

where

$$\left.\begin{aligned} A^* p &= \Big[\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathcal{D}^\alpha + 1 \Big] p \\ \frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) &= \sum_{|w|=0}^{\infty} (\mathcal{D}^w p(v)) \cos(n, x_i) \end{aligned}\right\} \tag{28}$$

The existence of a unique solution for the problem (23) - (27) on the cylinder Q can be proved using a constructive method. It is easy to notice that for given z_d and v, problem (23) - (27) can be solved backwards in time starting from $t = T$, i.e., first, solving (23) - (27) on the subcylinder Q_K and in turn on Q_{K-1}, etc. until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 1 (with an obvious change of variables) to problem (23) - (27) (with reversed sense of time, i.e., $t' = T - t$).

Lemma 2. Let the hypothesis of Theorem 1 be satisfied. Then, for given $z_d \in L^2(Q)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in H^{\infty,1}(Q)$ for the problem (23) - (27).

We simplify (22) using the adjoint equation (23) - (27). For this purpose setting $v = v_0$ in (23) - (27), multiplying both sides of (23) by $(y(v) - y(v_0))$ and then integrating over $\Omega \times (0,T)$ we get

$$\begin{aligned} &\lambda_1 \int_Q (y(v_0) - z_d)(y(v) - y(v_0)) dx dt \\ &= \int_Q \left(-\frac{\partial p(v_0)}{\partial t} + A^* p(v_0) \right) (y(v) - y(v_0)) dx dt \\ &= \int_Q p(v_0) \frac{\partial}{\partial t} (y(v) - y(v_0)) dx dt + \\ &+ \int_Q A^* p(v_0)(y(v) - y(v_0)) dx dt \end{aligned} \tag{29}$$

The second integral on the right-hand side of (29), in view of Green's formula, can be expressed as

$$\begin{aligned} &\int_Q A^* p(v_0)(y(v) - y(v_0)) dx dt \\ &= \int_Q p(v_0) A (y(v) - y(v_0)) \mathrm{dxdt} + \\ &+ \int_0^T \int_\Gamma p(v_0) \left(\frac{\partial y(v)}{\partial \eta_A} - \frac{\partial y(v_0)}{\partial \eta_A} \right) d\Gamma dt + \\ &- \int_0^T \int_\Gamma \frac{\partial p(v_0)}{\partial \eta_{A^*}} (y(v) - y(v_0)) d\Gamma dt \end{aligned} \tag{30}$$

Using the boundary condition (8), the second component on the right-hand side of (30) can be written as

$$\begin{aligned} &\int_0^T \int_\Gamma p(v_0) \left(\frac{\partial y(v)}{\partial \eta_A} - \frac{\partial y(v_0)}{\partial \eta_A} \right) d\Gamma dt = \\ &= \int_0^T \int_\Gamma p(x,t;v_0) \int_a^b c(x,t)(y(x,t-h;v) + \\ &- y(x,t-h;v_0)) dh) d\Gamma dt + \\ &+ \int_0^T \int_\Gamma p(x,t;v_0)(v - v_0) dx dt = \\ &= \int_0^T \int_\Gamma \int_a^b p(x,t;v_0) c(x,t) \cdot \\ &\cdot (y(x,t-h;v) - y(x,t-h;v_0)) dh\ d\Gamma dt + \\ &+ \int_0^T \int_\Gamma p(x,t;v_0)(v - v_0) dx dt = \\ &= \int_a^b \int_\Gamma \int_0^T p(x,t;v_0) c(x,t) \cdot \\ &\cdot (y(x,t-h;v) - y(x,t-h;v_0)) dt\ d\Gamma dh + \\ &+ \int_0^T \int_\Gamma p(x,t;v_0)(v - v_0) dx dt = \\ &= \int_a^b \int_\Gamma \int_{-h}^{T-h} p(x,t'+h;v_0) c(x,t'+h) \cdot \\ &\cdot (y(x,t';v) - y(x,t';v_0)) dt'\ d\Gamma dh + \\ &+ \int_0^T \int_\Gamma p(x,t;v_0)(v - v_0) dx dt = \\ &= \int_a^b \int_\Gamma \int_{-h}^{0} p(x,t'+h;v_0) c(x,t'+h) \cdot \end{aligned}$$

$$\cdot(y(x,t';v)-y(x,t';v_0))dt'\ d\Gamma dh +$$

$$+\int_a^b\int_\Gamma\int_0^{T-b} p(x,t'+h;v_0)c(x,t'+h)\cdot$$

$$\cdot(y(x,t';v)-y(x,t';v_0))dt'\ d\Gamma dh +$$

$$+\int_a^b\int_\Gamma\int_{T-b}^{T-h} p(x,t'+h;v_0)c(x,t'+h)\cdot$$

$$\cdot(y(x,t';v)-y(x,t';v_0))dt'\ d\Gamma dh +$$

$$+\int_0\int_\Gamma p(x,t;v_0)(v-v_0)dxdt =$$

$$=\int_a^b\int_\Gamma\int_{-h}^{0} p(x,t'+h;v_0)c(x,t'+h)\cdot$$

$$\cdot(y(x,t';v)-y(x,t';v_0))dt'\ d\Gamma dh +$$

$$+\int_a^b\int_\Gamma\int_0^{T-b} p(x,t'+h;v_0)c(x,t'+h)\cdot$$

$$\cdot(y(x,t';v)-y(x,t';v_0))dt'\ d\Gamma dh +$$

$$+\int_a^{T-t}\int_\Gamma\int_{T-b}^{T-a} p(x,t'+h;v_0)c(x,t'+h)\cdot$$

$$\cdot(y(x,t';v)-y(x,t';v_0))dt'\ d\Gamma dh +$$

$$+\int_0\int_\Gamma p(x,t;v_0)(v-v_0)dxdt \qquad (31)$$

The last component in (30) can be rewritten as

$$\int_0^T\int_\Gamma \frac{\partial p(v_0)}{\partial\eta_{A^*}}(y(v)-y(v_0))d\Gamma dt =$$

$$=\int_0^{T-b}\int_\Gamma \frac{\partial p(v_0)}{\partial\eta_{A^*}}(y(v)-y(v_0))d\Gamma dt +$$

$$+\int_{T-b}^{T-a}\int_\Gamma \frac{\partial p(v_0)}{\partial\eta_{A^*}}(y(v)-y(v_0))d\Gamma dt$$

$$+\int_{T-a}^{T}\int_\Gamma \frac{\partial p(v_0)}{\partial\eta_{A^*}}(y(v)-y(v_0))d\Gamma dt \qquad (32)$$

Substituting (31), (32) into (30) and then (30) into (29), we obtain

$$\lambda_1\int_Q (y(v_0)-z_d)(y(v)-y(v_0))dxdt =$$

$$=\int_Q p(v_0)\left(\frac{\partial}{\partial t}+A\right)(y(v)-y(v_0))dxdt +$$

$$+\int_a^b\int_\Gamma\int_{-h}^{0} c(x,t+h)p(x,t+h;v_0)\cdot$$

$$\cdot(y(x,t;v)-y(x,t;v_0))dt\ d\Gamma dh +$$

$$+\int_a^b\int_\Gamma\int_0^{T-b} c(x,t+h)p(x,t+h;v_0)\cdot$$

$$\cdot(y(x,t;v)-y(x,t;v_0))dt\ d\Gamma dh +$$

$$+\int_a^{T-t}\int_\Gamma\int_{T-b}^{T-a} c(x,t+h)p(x,t+h;v_0)\cdot$$

$$\cdot(y(x,t;v)-y(x,t;v_0))dt\ d\Gamma dh +$$

$$+\int_0^T\int_\Gamma p(x,t;v_0)(v-v_0)d\Gamma dt +$$

$$-\int_0^{T-b}\int_\Gamma \frac{\partial p(v_0)}{\partial\eta_{A^*}}(y(x,t;v)-y(x,t;v_0))d\Gamma dt +$$

$$-\int_{T-b}^{T-a}\int_\Gamma \frac{\partial p(v_0)}{\partial\eta_{A^*}}(y(x,t;v)-y(x,t;v_0))d\Gamma dt +$$

$$-\int_{T-a}^{T}\int_\Gamma \frac{\partial p(v_0)}{\partial\eta_{A^*}}(y(x,t;v)-y(x,t;v_0))d\Gamma dt =$$

$$=\int_0^T\int_\Gamma p(x,t;v_0)(v-v_0)d\Gamma dt \qquad (33)$$

Substituting (33) into (22) we obtain

$$\int_\Sigma (p(v_0)+\lambda_2 N v_0)(v-v_0)d\Gamma dt + \qquad (34)$$

$$+\lambda_3\int_\Sigma |v|d\Gamma dt-\lambda_3\int_\Sigma |v_0|d\Gamma dt \ge 0 \quad \forall v\in U_{ad}$$

Theorem 2. For the problem (6) - (9) with the performance functional (19) with $z_d \in L^2(Q)$ and $\lambda_2 > 0$, $\lambda_3 > 0$ and with constraints on controls (20), there exists a unique optimal control v_0 which satisfies the maximum condition (34).

We can also consider an analogous optimal control problem where the performance functional is given by

$$\hat{I}(v)=\lambda_1\int_\Sigma \left|y(v)|_\Sigma - z_{\Sigma d}\right|^2 d\Gamma dt +$$

$$+\lambda_2\int_\Sigma (Nv)v d\Gamma dt + 2\lambda_3\int_\Sigma |v|d\Gamma dt \qquad (35)$$

From the Theorem 1 and the Trace Theorem (Lions and Magenes (1972), Vol.2,p. 9), for each $v \in L^2(\Sigma)$, there exists a unique solution $y \in H^{\infty,1}(Q)$ with $y|_\Sigma \in H^{\infty,1}(\Sigma) \subset L^2(\Sigma)$. Thus $\hat{I}(v)$ is well-defined. Then, the optimal control v_0 is characterized by

$$\lambda_1 \int\limits_{\Sigma} (y(v_0)|_{\Sigma} - z_{\Sigma d})(y(v)|_{\Sigma} - y(v_0)|_{\Sigma}) d\Gamma dt + +$$
$$+\lambda_2 \int\limits_{\Sigma} (Nv_0)(v - v_0) d\Gamma dt + \lambda_3 \int\limits_{\Sigma} |v| d\Gamma dt + +$$
$$-\lambda_3 \int\limits_{\Sigma} |v_0| d\Gamma dt \geq 0 \qquad \forall v \in U_{ad} \tag{36}$$

We introduce the following adjoint equation

$$-\frac{\partial p(v_0)}{\partial t} + A^* p(v_0) = 0 \quad x \in \Omega, \ t \in (0, T) \tag{37}$$

$$p(x, T; v_0) = 0 \quad x \in \Omega \tag{38}$$

$$\frac{\partial p(v_0)}{\partial \eta_{A^*}} = \int\limits_a^b c(x, t+h) p(x, t+h; v_0) dh + \tag{39}$$
$$+\lambda_1 (y(v_0)|_{\Sigma} - z_{\Sigma d}) \quad x \in \Gamma, t \in (0, T-b)$$

$$\frac{\partial p(v_0)}{\partial \eta_{A^*}} = \int\limits_a^{T-t} c(x, t+h) p(x, t+h; v_0) dh + \tag{40}$$
$$+\lambda_1 (y(v_0)|_{\Sigma} - z_{\Sigma d}) \quad x \in \Gamma, t \in (T-b, T-a)$$

$$\frac{\partial p(v_0)}{\partial \eta_{A^*}} = \lambda_1 (y(v_0)|_{\Sigma} - z_{\Sigma d}) \quad x \in \Gamma, t \in (T-a, T) \tag{41}$$

Using the Theorem 1 the following lemma can be proved.

Lemma 3. Let the hypothesis of Theorem 1 be satisfied. Then, for given $z_{\Sigma d} \in L^2(\Sigma)$ and any $v_0 \in L^2(\Sigma)$, there exists a unique solution $p(v_0) \in H^{\infty,1}(Q)$ to the problem (37) - (41).

In this case the condition (36) can be also rewritten in the form (34). The following theorem is now fulfilled.

Theorem 3. For the problem (6) - (9) with the performance functional (35) with $z_{\Sigma d} \in L^2(\Sigma)$ and $\lambda_2 > 0$, $\lambda_3 > 0$ and with constraints on control (20), there exists a unique optimal control v_0 which satisfies the maximum condition (34).

Remark 1. The uniqueness of the optimal control follows from the strict convexity of performance functionals (19) and (35) with $v = v^+ - v^-$ and $|v| = v^+ + v^-$ where $v^+ = \dfrac{|v| + v}{2}$ and $v^- = \dfrac{|v| - v}{2}$.

We must notice that the conditions of optimality derived above (Theorems 2 and 3) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on controls). This results from the following: the determining of the function $p(v_0)$ in the maximum condition from the adjoint equation is possible if and only if we know y_0 which corresponds to the control v_0. These mutual connections make the practical use of the derived optimization formulas difficult. Therefore we resign from the exact determining of the optimal control and we use approximation methods (Kowalewski (1988), Kowalewski and Duda (1992), Wong (1987)).

4. CONCLUSIONS

The results presented in the paper can be treated as a generalization of the results obtained in Wang (1975) onto the case of infinite order parabolic optimal control problems with non-differentiable performance functional and time delays given in the integral form.

In this paper we have considered the optimal parabolic systems with the Neumann boundary conditions involving time delays in the integral form.

We can also derived conditions of optimality for a more complex case of such parabolic systems with the Dirichlet boundary conditions.

Finally, we can consider similar optimal control problems for infinite order hyperbolic systems with time delays given in the integral form.

The ideas mentioned above will be developed in forthcoming papers.

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